Strassen's Matrix Multiplication
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[partially based on slides by Prof. Welch]

Matrix Multiplication

Consider two n x n matrices A and B

Recall that the matrix product $C = AB$ of two n x n matrices is defined as the n x n matrix that has the coefficient

$$c_{kl} = \Sigma_m \, a_{km} \, b_{ml}$$

in row k and column l, where the sum ranges over the integers from 1 to n; the scalar product of the kth row of a with the lth column of B.

The straightforward algorithm uses $O(n^3)$ scalar operations.

Can we do better?

Idea: Use Divide and Conquer

The divide and conquer paradigm is important general technique for designing algorithms. In general, it follows the steps:

- divide the problem into subproblems

- recursively solve the subproblems

- combine solutions to subproblems to get solution to original problem

Divide-and-Conquer

Let write the product A B = C as follows:

A	A
A	A

×

B	B
B	B

=

A	A
A	A

- Divide matrices A and B into four submatrices each

- We have 8 smaller matrix multiplications and 4 additions. Is it faster?

Divide-and-Conquer

Let us investigate this recursive version of the matrix multiplication.

Since we divide A, B and C into 4 submatrices each, we can compute the resulting matrix C by

- 8 matrix multiplications on the submatrices of A and B,

- plus $\Theta(n^2)$ scalar operations

Divide-and-Conquer

- Running time of recursive version of straightfoward algorithm is

    $T(n) = 8T(n/2) + \Theta(n^2)$ and $T(2) = \Theta(1)$

    where $T(n)$ is running time on an n x n matrix

- Master theorem gives us:

    $T(n) = \Theta(n^3)$

- Can we do fewer recursive calls (fewer multiplications of the n/2 x n/2 submatrices)?

Strassen's Matrix Multiplication

$$A \times B = C \quad .$$

A	A
A	A

$\times$

B	B
B	B

$=$

C	
C	C

$P_1 = (A_{11} + A_{22})(B_{11} + B_{22})$

$P_2 = (A_{21} + A_{22}) * B_{11}$

$P_3 = A_{11} * (B_{12} - B_{22})$

$P_4 = A_{22} * (B_{21} - B_{11})$

$P_5 = (A_{11} + A_{12}) * B_{22}$

$P_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$

$P_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$

$C_{11} = P_1 + P_4 - P_5 + P_7$

$C_{12} = P_3 + P_5$

$C_{21} = P_2 + P_4$

$C_{22} = P_1 + P_3 - P_2 + P_6$

Strassen's Matrix Multiplication

- Strassen found a way to get all the required information with only 7 matrix multiplications, instead of 8.

- Recurrence for new algorithm is

  - $T(n) = 7T(n/2) + \Theta(n^2)$

Solving the Recurrence Relation

Applying the Master Theorem to

$T(n) = a\, T(n/b) + f(n)$

with a=7, b=2, and $f(n) = \Theta(n^2)$.

Since $f(n) = O(n^{\log_b(a) - \varepsilon}) = O(n^{\log_2(7) - \varepsilon})$,

case a) applies and we get

$T(n) = \Theta(n^{\log_b(a)}) = \Theta(n^{\log_2(7)}) = O(n^{2.81})$.

Discussion of Strassen's Algorithm

- Not always practical

  - constant factor is larger than for naïve method

  - specially designed methods are better on sparse matrices

  - issues of numerical (in)stability

  - recursion uses lots of space

- Not the fastest known method

  - Fastest known is $O(n^{2.3727})$ [Winograd-Coppersmith algorithm improved by V. Williams]

  - Best known lower bound is $\Omega(n^2)$