# Finding the Second Largest Element

Andreas Klappenecker

# Problem

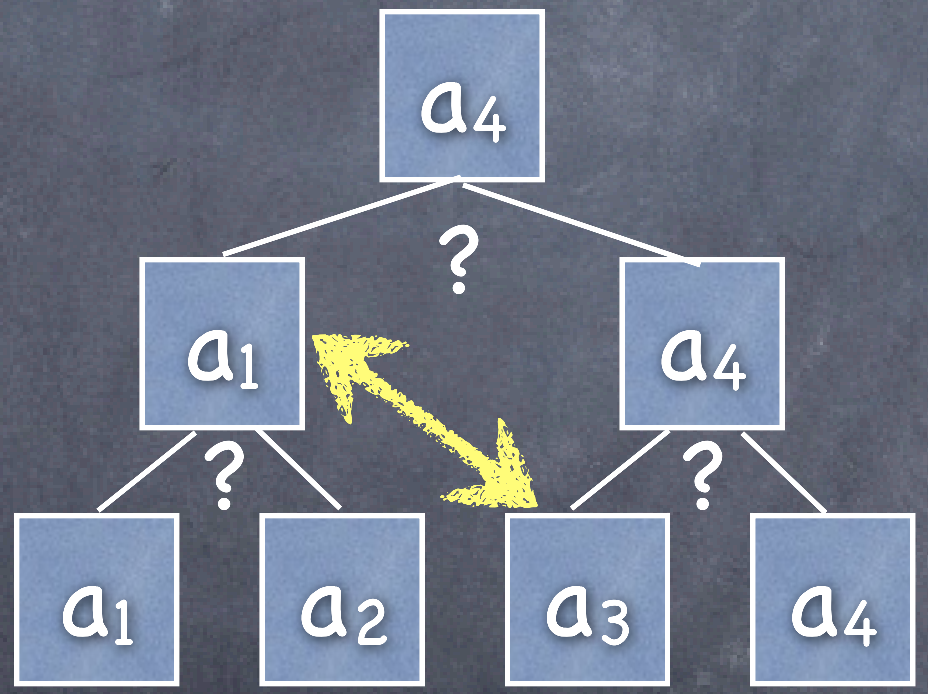
Given a set of n elements from a totally ordered domain, our goal is to find the second largest element $m_2$.

How many queries are needed to determine $m_2$?

# Upper Bound

We can compare the elements pairwise in a tournament style.

If $x < y$, then we say that $y$ wins.




- If an element was never compared to the largest element, then it cannot be second largest (e.g. $a_2$).
- Find second largest among the one's who have lost to the largest.
- So $\le (n-1) + \lceil \lg n \rceil - 1$ comparisons

# Lower Bound

Any algorithm to determine the second largest element of a totally ordered set n elements needs at least $(n-2) + \lceil \lg n \rceil$ comparisons in the worst case.

# Lower Bound

Let $m_1$ be the largest element and $m_2$ the second largest element.

An algorithm to determine $m_2$ needs to find the largest element $m_1$ for otherwise an adversary would be able to exchange $m_1$ for $m_2$.

Furthermore, the n-2 elements below $m_2$ must be identified by the algorithm, meaning that they must have lost in comparison to $m_2$ or some element below $m_2$. This means that there are n-2 comparisons that do not involve $m_1$.

It remains to show that an adversary can force any algorithm to do at least $\lceil \lg n \rceil$ comparisons with the largest element $m_1$.

# Adversary 1

Our goal is to show that an algorithm Z needs to make lg n or more comparisons with the largest element.

We construct an adversary that answers comparisons "Is a <= b?" consistent with a total order of the n elements.

For each element x, we let K(x) denote the set of elements y known to Z that satisfy y <= x. Initially K(x) = {x}.

The adversary uses previous query history of Z and K(a) and K(b) to create answer for questions such as "Is a <= b".

# Adversary 2

The adversary behaves as follows:

- If “Is a <= b?” was asked before, give same answer.

- If “Is a <= b?” was not asked before, then answer

  - yes if |K(a)| <= |K(b)|. Update $K(b) := K(a) \cup K(b)$
  - no, if |K(a)| > |K(b)|. Update $K(a) := K(a) \cup K(b)$

# Adversary 3

Let S be the totally ordered domain of n elements.

- At the beginning $|K(a)| = 1$ holds for all a in S.

- For each query involving a, $|K(a)|$ can at most double.

- Since Z needs to determine largest element, $|K(m_1)| = n$ must hold at the end.

- The number k of queries involving $m_1$ satisfies $2^k >= n$, so $k >= \lg n$

and since k must be an integer, we have $k >= \lceil \lg n \rceil$.

# Conclusions

Any algorithm to determine the second largest element of a totally ordered set n elements needs at least $(n-2) + \lceil \lg n \rceil$ comparisons in the worst case.

We have given an optimal algorithm that attains this lower bound.