Explore Euler's Constant e

Form a group of four people to discuss and explore the following questions:

1. Show that $(1+1/n)^n$ is monotonically increasing.
2. Show that e is the limit of $(1+1/n)^n$ as $n \to \infty$.
   [Hint: Use $(1+\frac{1}{n})^n \le e \le (1+\frac{1}{n})^{n+1}$.]
3. Bound the probability that a 10 digit number is prime.

One person should be the scribe, and the other three should each take the lead on one of the questions.

Solution:

1. We would like to show that

$$(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1} \tag{1}$$

holds for all $n \ge 1$. Recall that the arithemetic mean of $n+1$ real numbers $x_1, \ldots, x_{n+1}$ is lower bounded by the geometric mean,

$$\left(\frac{x_1+\cdots+x_{n+1}}{n+1}\right) \ge \sqrt[n+1]{x_1 x_2 \cdots x_{n+1}},$$

and equality holds if and only if $x_1 = x_2 = \cdots = x_{n+1}$. Equivalently, we have

$$\left(\frac{x_1+\cdots+x_{n+1}}{n+1}\right)^{n+1} \ge x_1 x_2 \cdots x_{n+1}, \tag{2}$$

and equality holds if and only if $x_1 = x_2 = \cdots = x_{n+1}$.

Let $A_{n+1} = 1 \cdot (1+\frac{1}{n})^n$ be the product of n terms $(1+1/n)$ and the term 1, and $B_{n+1} = (1+\frac{1}{n+1})^{n+1}$ be the product of $n+1$ terms $1+1/(n+1)$. The arithmetic mean of the terms in A_{n+1} and B_{n+1} are both the same, namely $(n+2)/(n+1)$. Therefore, using (2) twice for both A_{n+1} and B_{n+1}, we have

$$B_{n+1} \overset{(\le)}{=} \left(\frac{n+2}{n+1}\right)^{n+1} > A_{n+1},$$

which proves (1).

Remark: You can use many other methods to prove the same thing. For example, you can show that $B_{n+1}/A_{n+1} > 1$ by simplifying the left hand side.

2. By using the inequality on the right of the hint and dividing it by $1+1/n$, we get

$$\frac{e}{1+1/n} \le \left(1+\frac{1}{n}\right)^n \le e.$$

Taking limits, we find that

$$e = \lim_{n\to\infty} \frac{e}{1+1/n} \leq \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \leq \lim_{n\to\infty} e = e,$$

so by the pinching theorem, we can conclude that the limit must be

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n,$$

as claimed.

3. See slides.

What skills did we train?

1. Proving inequalities. You need that skill frequently when making estimates of running time.

2. Limits. You need knowledge about limits when dealing with asymptotic notions such as Big Oh and Big Omega.

3. Probabilistic analysis. You need knowledge about the density of primes in applications in cryptography and the techniques apply to randomized algorithms.

Make sure that you complete your own approach (or try to understand where your team went wrong).