Divide and Conquer

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[based on slides by Prof. Welch]

Divide and Conquer Paradigm

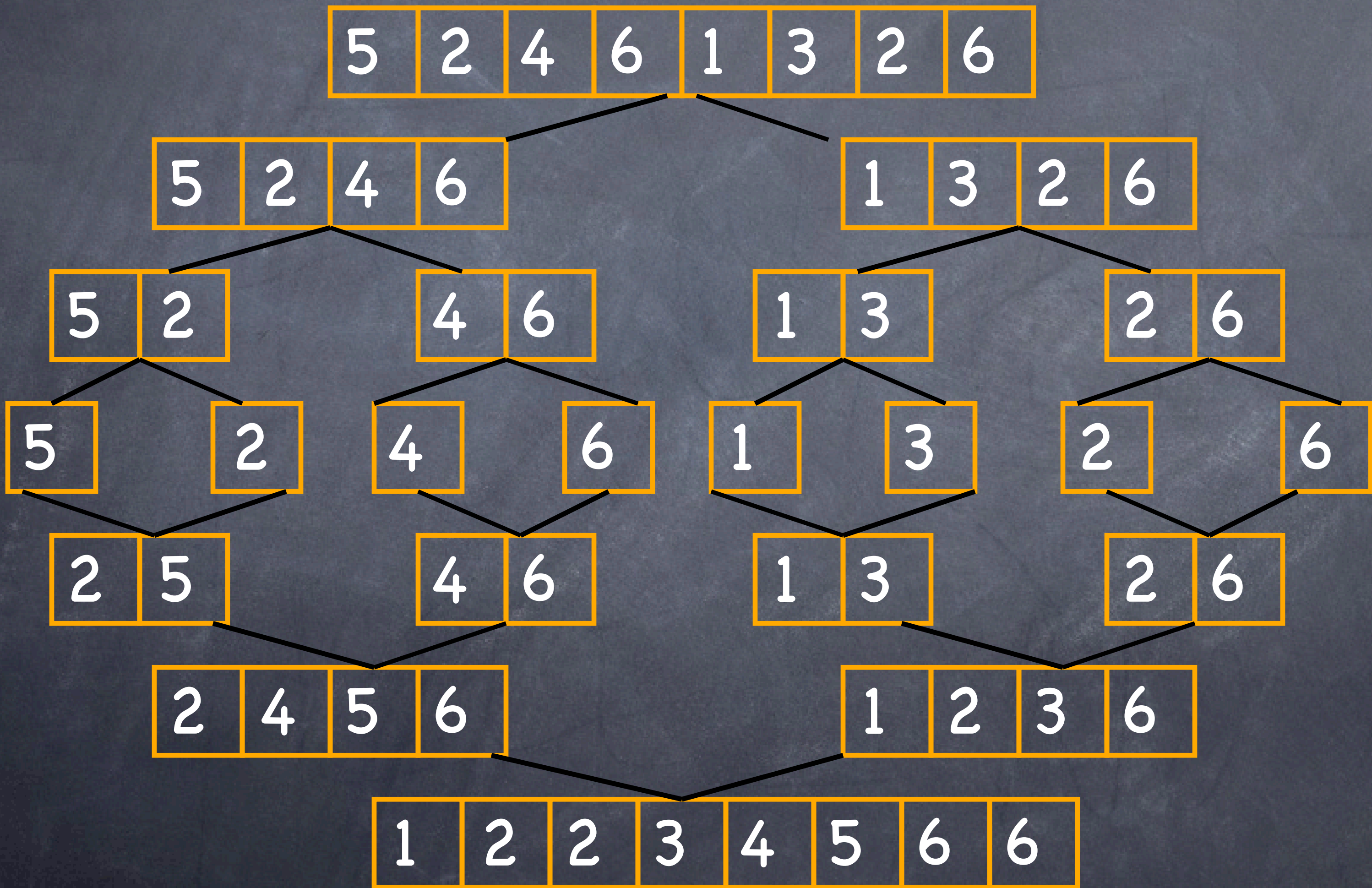
- An important general technique for designing algorithms:
  - divide problem into subproblems
  - recursively solve subproblems
  - combine solutions to subproblems to get solution to original problem
- Use recurrences to analyze the running time of such algorithms

Mergesort

Example: Mergesort

- DIVIDE the input sequence in half
- RECURSIVELY sort the two halves
  - basis of the recursion is sequence with 1 key
- COMBINE the two sorted subsequences by merging them

Mergesort Example



Mergesort Animation

- http://ccl.northwestern.edu/netlogo/models/run.cgi?MergeSort.862.378

Recurrence Relation for Mergesort

- Let $T(n)$ be worst case time on a sequence of n keys
- If $n = 1$, then $T(n) = \Theta(1)$ (constant)
- If $n > 1$, then $T(n) = 2\ T(n/2) + \Theta(n)$
  - two subproblems of size $n/2$ each that are solved recursively
  - $\Theta(n)$ time to do the merge

Recurrence Relations

How To Solve Recurrences

- Ad hoc method:
  - expand several times
  - guess the pattern
  - can verify with proof by induction
- Master theorem
  - general formula that works if recurrence has the form $T(n) = aT(n/b) + f(n)$
    - a is number of subproblems
    - n/b is size of each subproblem

Master Theorem

Consider a recurrence of the form

$T(n) = a\,T(n/b) + f(n)$

with a>=1, b>1, and f(n) eventually positive.

a) If $f(n) = O(n^{\log_b(a)-\varepsilon})$, then $T(n)=\Theta(n^{\log_b(a)})$.

b) If $f(n) = \Theta(n^{\log_b(a)})$, then $T(n)=\Theta(n^{\log_b(a)} \log(n))$.

c) If $f(n) = \Omega(n^{\log_b(a)+\varepsilon})$ and f(n) is regular, then $T(n)=\Theta(f(n))$

[f(n) regular iff eventually $af(n/b) <= cf(n)$ for some constant $c<1$]

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Now go back and memorize the theorem!

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Déjà vu: Master Theorem

Consider a recurrence of the form

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with a>=1, b>1, and f(n) eventually positive.

a) If $f(n) = O(n^{\log_b(a)-\varepsilon})$, then $T(n)=\Theta(n^{\log_b(a)})$.

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c) If $f(n) = \Omega(n^{\log_b(a)+\varepsilon})$ and f(n) is regular, then $T(n)=\Theta(f(n))$

[f(n) regular iff eventually af(n/b)<= cf(n) for some constant c<1]

Nothing is perfect...

The Master theorem does not cover all possible cases. For example, if

$$f(n) = \Theta(n^{\log_b(a)} \log n),$$

then we lie between cases 2) and 3), but the theorem does not apply.

There exist better versions of the Master theorem that cover more cases, but these are even harder to memorize.

Idea of the Proof

Let us iteratively substitute the recurrence:

$$
\begin{aligned}
T(n) &= aT(n/b) + f(n) \\
&= a(aT(n/b^2)) + f(n/b)) + bn \\
&= a^2T(n/b^2) + af(n/b) + f(n) \\
&= a^3T(n/b^3) + a^2 f(n/b^2) + af(n/b) + f(n) \\
&= \ldots \\
&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\
&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)
\end{aligned}
$$

Idea of the Proof

Thus, we obtained

$T(n) = n^{\log_b(a)}\, T(1) + \sum a^i f(n/b^i)$

The proof proceeds by distinguishing three cases:

1) The first term in dominant: $f(n) = O(n^{\log_b(a)-\varepsilon})$

2) Each part of the summation is equally dominant: $f(n) = \Theta(n^{\log_b(a)})$

3) The summation can be bounded by a geometric series: $f(n) = \Omega(n^{\log_b(a)+\varepsilon})$ and the regularity of f is key to make the argument work.

Further Divide and Conquer Examples

Additional D&C Algorithms

- binary search
  - divide sequence into two halves by comparing search key to midpoint
  - recursively search in one of the two halves
  - combine step is empty
- quicksort
  - divide sequence into two parts by comparing pivot to each key
  - recursively sort the two parts
  - combine step is empty

Additional D&C applications

- computational geometry
  - finding closest pair of points
  - finding convex hull
- mathematical calculations
  - converting binary to decimal
  - integer multiplication
  - matrix multiplication
  - matrix inversion
  - Fast Fourier Transform

Strassen's Matrix Multiplication

Matrix Multiplication

- Consider two n by n matrices A and B
- Definition of AxB is n by n matrix C whose (i,j)-th entry is computed like this:
  - consider row i of A and column j of B
  - multiply together the first entries of the rown and column, the second entries, etc.
  - then add up all the products
- Number of scalar operations (multiplies and adds) in straightforward algorithm is $O(n^3)$.
- Can we do it faster?

Divide-and-Conquer

$$A \times B = C$$

$$\begin{array}{|c|c|}\hline A_0 & A_1 \\ \hline A_2 & A_3 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline B_0 & B_1 \\ \hline B_2 & B_3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline A_0 \times B_0 + A_1 \times & A_0 \times B_1 + A_1 \times \\ \hline A_2 \times B_0 + A_3 \times & A_2 \times B_1 + A_3 \times \\ \hline \end{array}$$

- Divide matrices A and B into four submatrices each
- We have 8 smaller matrix multiplications and 4 additions. Is it faster?

Divide-and-Conquer

Let us investigate this recursive version of the matrix multiplication.

Since we divide A, B and C into 4 submatrices each, we can compute the resulting matrix C by

- 8 matrix multiplications on the submatrices of A and B,
- plus $\Theta(n^2)$ scalar operations

Divide-and-Conquer

Divide-and-Conquer

- Running time of recursive version of straightfoward algorithm is
  - $T(n) = 8T(n/2) + \Theta(n^2)$
  - $T(2) = \Theta(1)$

  where $T(n)$ is running time on an $n \times n$ matrix

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- Master theorem gives us:

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- Can we do fewer recursive calls (fewer multiplications of the $n/2 \times n/2$ submatrices)?

Strassen's Matrix

A × B = C .

$$\begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix} \times \begin{pmatrix} B_0 & B_1 \\ B_2 & B_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$P_1 = (A_{11} + A_{22})(B_{11} + B_{22})$$

$$P_2 = (A_{21} + A_{22}) * B_{11}$$

$$P_3 = A_{11} * (B_{12} - B_{22})$$

$$P_4 = A_{22} * (B_{21} - B_{11})$$

$$P_5 = (A_{11} + A_{12}) * B_{22}$$

$$P_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$$

$$P_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$$

$$C_{11} = P_1 + P_4 - P_5 + P_7$$

$$C_{12} = P_3 + P_5$$

$$C_{21} = P_2 + P_4$$

$$C_{22} = P_1 + P_3 - P_2 + P_6$$

Strassen's Matrix Multiplication

- Strassen found a way to get all the required information with only 7 matrix multiplications, instead of 8.
- Recurrence for new algorithm is
  - $T(n) = 7T(n/2) + \Theta(n^2)$

Solving the Recurrence Relation

Applying the Master Theorem to

$T(n) = a\ T(n/b) + f(n)$

with a=7, b=2, and $f(n)=\Theta(n^2)$.

Since $f(n) = O(n^{\log_b(a)-\varepsilon}) = O(n^{\log_2(7)-\varepsilon})$,

case a) applies and we get

$T(n)= \Theta(n^{\log_b(a)}) = \Theta(n^{\log_2(7)}) = O(n^{2.81})$.

Discussion of Strassen's Algorithm

- Not always practical
  - constant factor is larger than for naïve method
  - specially designed methods are better on sparse matrices
  - issues of numerical (in)stability
  - recursion uses lots of space
- Not the fastest known method
  - Fastest known is $O(n^{2.3727})$ [Winograd-Coppersmith algorithm improved by V. Williams]
  - Best known lower bound is $\Omega(n^2)$

Fast Integer Multiplication

Integer Multiplication

Elementary school algorithm (in binary)

101001 = 41

× 101010 = 42

1010100

1010100

+ 1010100

11010111010 = 1722

Integer Multiplication

Elementary school algorithm (in binary)

101001 = 41

× 101010 = 42

1010100

1010100

+ 1010100

11010111010 = 1722

Scan second number from right to left. Whenever you see a 1, add the first number to the result shifted by the appropriate number of bits.

Integer Multiplication

- The multiplication of two n bits numbers takes $\Omega(n^2)$ time using the elementary school algorithm.
- Can we do better?
- Kolmogorov conjectured in one of his seminars that one cannot, but was proved wrong by Karatsuba.

Divide and Conquer

- Let's split the two integers X and Y into two parts: their most significant part and their least significant part.
- $X = 2^{n/2}A + B$ (where A and B are n/2 bit integers)
- $Y = 2^{n/2}C + D$ (where C and D are n/2 bit integers)
- $XY = 2^{n}AC + 2^{n/2}BC + 2^{n/2}AD + BD$.

How Did We Do?

- Multiplication by 2^x can be done in hardware with very low cost (just a shift).
- We can apply this algorithm recursively:
- We replaced one multiplication of n bits numbers by four multiplications of n/2 bits numbers and 3 shifts and 3 additions
- $T(n) = 4T(n/2) + cn$

Solve the Recurrence

- $T(n) = 4T(n/2) + cn$
- By the Master theorem, $g(n) = n^{\log_2(4)} = n^2$.
- Since $cn = O(n^{2-\text{epsilon}})$, we can conclude that $T(n) = \Theta(n^2)$.

How can we do better?

- Suppose that we are able to reduce the number of multiplications from 4 to 3, allowing for more additions and shifts (but still a constant number).
- Then $T(n) = 3T(n/2) + dn$
- Master theorem: $dn = O(n^{\log2(3)-\text{epsilon}})$, so we get $T(n) = O(n^{\log2(3)}) = O(n^{1.585})$.

Karatsuba's Idea

Let's rewrite

$$XY = 2^n AC + 2^{n/2} BC + 2^{n/2} AD + BD$$

in the form

$$XY = (2^n - 2^{n/2})AC + 2^{n/2}(A+B)(C+D) + (1-2^{n/2})BD.$$

Done! Wow!

Summary

Split input X into two parts A and B such that $X = 2^{n/2}A + B$.

Split input Y into two parts C and D such that

$Y = 2^{n/2}C + D$.

Then calculate AC, (A+B)(C+D), BD.

Copy and shift the results, and add/subtract:

$$XY = (2^n - 2^{n/2})AC + 2^{n/2}(A+B)(C+D) + (1-2^{n/2})BD.$$