Amortized Analysis

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[partially based on the slides of Prof. Welch]

Analyzing Calls to a Data

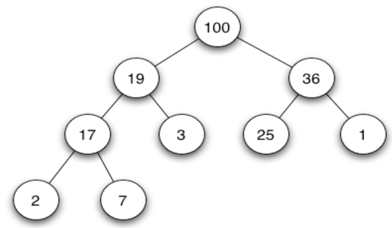
- Some algorithms involve repeated calls to one or more data structures.
- When analyzing the running time of an algorithm, one needs to sum up the time spent in all the calls to the data structure.
- Problem: If different calls take different times, how can we accurately calculate the total time?

Max-Heap

A max-heap is an nearly complete binary tree

(i.e., all levels except the deepest level are completely filled and the last level is filled from the left)

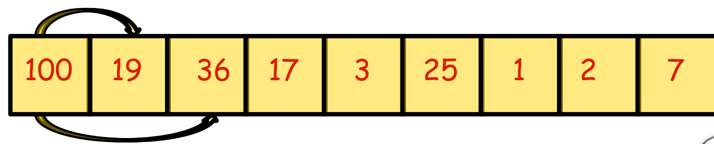
satisfying the heap property: if B is a child of a node A, then key[A] >= key[B].

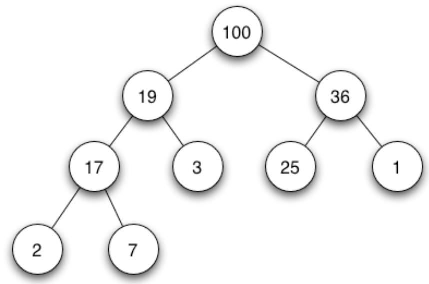


[Picture courtesy of Wikipedia.]

Heap Implementation

We can store a heap in an array:





If the array is indexed a[1..n],

then a[i] has children a[2i] and a[2i+1]:

a[1] has children a[2], a[3],

a[2] has children a[4], a[5],

a[3] has children a[6], a[7], ...

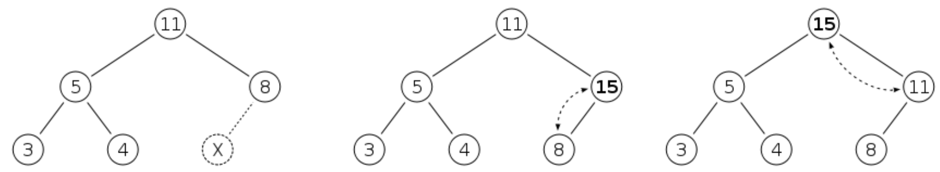
Adding an Element to a Heap

An element can be added to the heap as follows:

1. Add the element on the bottom level of the heap.
2. Compare the added element with its parent; if they are in the correct order, stop.
3. If not, swap the element with its parent and return to the previous step.

Adding an Element: Example

Adding 15 to a max-heap. Insert at position x, compare with parent, swap, compare with parent, swap.



What is the time-complexity of adding an element to a heap with n elements?

Constructing a Heap

Let us form a heap of n elements from scratch.

First idea:

- Use n times add to form the heap.
- Each addition to the heap operates on a heap with at most n elements.
- Adding to a heap with n elements takes O(log n) time
- Total time spent doing the n insertions is O (n log n) time

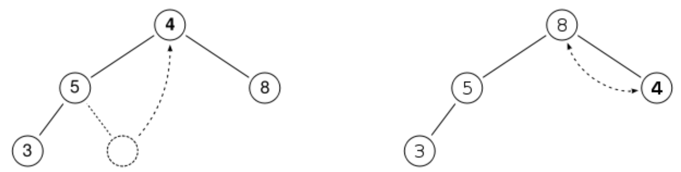
Constructing a Heap (2)

Two question arise:

- Does our analysis overestimate the time?
  The different insertions take different amounts of time, and many are on smaller heaps. (=> leads to amortized analysis)
- Is this the optimal way to create a heap?
  Perhaps simply adding n times is not the best way to form a heap.

Deleting the Maximal Element

Deleting the maximal element from a max-heap starts by replacing it with the last element from the lowest level. Then restore the heap property (using Max-Heapify) by swapping with largest child, and repeat same process on the next level, etc.



Constructing a Heap (3)

Second idea:

Place elements in an array, interpret as a binary tree. Look at subtrees at height h (measured from lowest level). If these trees have been heapified, then subtrees at height h+1 can be heapified by sending their roots down.

Initially, the trees at height 0 are all heapified.

Constructing a Heap (4)

Array of length n. Number of nodes at height h

is at most floor($n/2^{h+1}$). Cost to heapify a tree at height h+1 if all subtrees have been heapified: O(h) swaps. Total cost:

$$\begin{aligned}\sum_{h=0}^{\lceil \lg n \rceil} \frac{n}{2^{h+1}} O(h) &= O\left(n \sum_{h=0}^{\lceil \lg n \rceil} \frac{h}{2^h}\right) \\ &\leq O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) \\ &= O(n)\end{aligned}$$

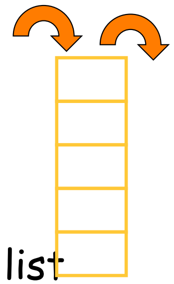
Amortized Analysis

Amortized Analysis

- Purpose is to accurately compute the total time spent in executing a sequence of operations on a data structure
- Three different approaches:
  - aggregate method: brute force
  - accounting method: assign costs to each operation so that it is easy to sum them up while still ensuring that the result is accurate
  - potential method: a more sophisticated version of the accounting method (omitted here)

Running Example #1:

- Operations are:
  - Push(S,x)
  - Pop(S)
  - Multipop(S,k) - pop the top k elements
- Implement with either array or linked list
  - time for Push is O(1)
  - time for Pop is O(1)
  - time for Multipop is O(min(|S|,k))



Running Example #2:

- Operation:
  - increment(A) - add 1 (initially 0)
- Implementation:
  - k-element binary array
  - use grade school ripple-carry algorithm

Aggregate Method

Aggregate Method

- Show that a sequence of n operations takes T(n) time
- We can then say that the amortized cost per operation is T(n)/n
- Makes no distinction between operation types

Augmented Stack:

- In a sequence of n operations, the stack never holds more than n elements.
- Thus, the cost of a multipop is $O(n)$
- Therefore, the worst-case cost of any sequence of n operations is $O(n^2)$.
- But this is an over-estimate!

Aggregate Method for

- Key idea: total number of pops (or multipops) in the entire sequence of operations is at most the total number of pushes

- Suppose that the maximum number of Push operations in the sequence is n.
- So time for entire sequence is O(n).
- Amortized cost per operation: O(n)/n = O(1).

Aggregate Method for k-Bit

- Worst-case time for an increment is O(k). This occurs when all k bits are flipped
- But in a sequence of n operations, not all of them will cause all k bits to flip:
  - bit 0 flips with every increment
  - bit 1 flips with every 2nd increment
  - bit 2 flips with every 4th increment …
  - bit k flips with every 2^k-th increment

Aggregate Method for k-Bit

- Total number of bit flips in n increment operations is
  - $n + n/2 + n/4 + \ldots + n/2^k < n(1/(1-1/2)) = 2n$
- So total cost of the sequence is $O(n)$.
- Amortized cost per operation is $O(n)/n = O(1)$.

Accounting Method

Accounting Method

- Assign a cost, called the "amortized cost", to each operation
- Assignment must ensure that the sum of all the amortized costs in a sequence is at least the sum of all the actual costs
  - remember, we want an upper bound on the total cost of the sequence

Accounting Method

- For each operation in the sequence:
  - if amortized cost > actual cost then store extra as a credit with an object in the data structure
  - if amortized cost < actual cost then use the stored credits to make up the difference
- Never allowed to go into the red! Must have enough credit saved up to pay for

Accounting Method vs.

- Aggregate method:
  - first analyze entire sequence
  - then calculate amortized cost per operation
- Accounting method:
  - first assign amortized cost per operation
  - check that they are valid (never go into the red)

Accounting Method for

- Assign these amortized costs:
  - Push - 2
  - Pop - 0
  - Multipop - 0
- For Push, actual cost is 1. Store the extra 1 as a credit, associated with the pushed element.
- Pay for each popped element (either from

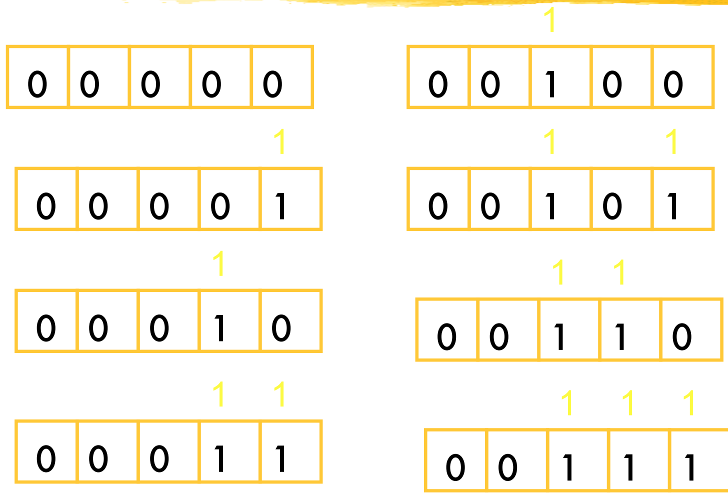
Accounting Method for

- There is always enough credit to pay for each operation (never go into red).
- Each amortized cost is $O(1)$
- So cost of entire sequence of n operations is $O(n)$.

Accounting Method for k-Bit

- Assign amortized cost for increment operation to be 2.
- Actual cost is the number of bits flipped:
  - a series of 1's are reset to 0
  - then a 0 is set to 1
- Idea: 1 is used to pay for flipping a 0 to 1. The extra 1 is stored with the

Accounting Method for k-Bit



Accounting Method for k-Bit

- All changes from 1 to 0 are paid for with previously stored credit (never go into red)
- Amortized time per operation is $O(1)$
- total cost of sequence is $O(n)$

Conclusions

Amortized Analysis allows one to estimate the cost of a sequence of operations on data structures.

The method is typically more accurate than worst case analysis when the data structure is dynamically changing.