Propositional Logic

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Propositions

A proposition is a declarative sentence that is either true or false (but not both).

Examples:

- College Station is the capital of the USA.
- There are fewer politicians in College Station than in Washington, D.C.
- 1+1=2
- 2+2=5

Propositional Variables

A variable that represents propositions is called a propositional variable.

For example: p, q, r, ...

[Propositional variables in logic play the same role as numerical variables in arithmetic.]

Propositional Logic

The area of logic that deals with propositions is called propositional logic.

In addition to propositional variables, we have logical connectives such as not, and, or, conditional, and biconditional.

Syntax of Propositional Logic

Approach

We are going to present the propositional logic as a formal language:

- we first present the syntax of the language

- then the semantics of the language.

[Incidentally, this is the same approach that is used when defining a new programming language. Formal languages are used in other contexts as well.]

Formal Languages

Let S be an alphabet.

We denote by S* the set of all strings over S, including the empty string.

A formal language L over the alphabet S is a subset of S*.

Syntax of Propositional Logic

Our goal is to study the language Prop of propositional logic. This is a language over the alphabet $\Sigma = S \cup X \cup B$, where

- $S = \{ a, a_0, a_1, \ldots, b, b_0, b_1, \ldots \}$ is the set of symbols,
- $X = \{\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow\}$ is the set of logical connectives,
- $B = \{ (\ ,\ ) \}$ is the set of parentheses.

We describe the language Prop using a grammar.

Grammar of Prop

$$\langle \text{formula} \rangle ::= \neg \langle \text{formula} \rangle$$

$$\mid (\langle \text{formula} \rangle \wedge \langle \text{formula} \rangle)$$

$$\mid (\langle \text{formula} \rangle \vee \langle \text{formula} \rangle)$$

$$\mid (\langle \text{formula} \rangle \oplus \langle \text{formula} \rangle)$$

$$\mid (\langle \text{formula} \rangle \rightarrow \langle \text{formula} \rangle)$$

$$\mid (\langle \text{formula} \rangle \leftrightarrow \langle \text{formula} \rangle)$$

$$\mid \langle \text{symbol} \rangle$$

Example

Using this grammar, you can infer that

$((a \oplus b) \vee c)$

$((a \rightarrow b) \leftrightarrow (\neg a \vee b))$

both belong to the language Prop, but

$((a \rightarrow b) \vee c$

does not, as the closing parenthesis is missing.

Meaning?

So far, we have introduce the syntax of propositional logic. Thus, we know that

$$((a \to b) \leftrightarrow (\neg a \vee b))$$

is a valid formula in propositional logic. However, we do not know yet the meaning of this formula. We need to give an unambiguous meaning to every formula in Prop.

Semantics

Formation Tree

Each logical connective is enclosed in parentheses, except for the negation connective ¬. Thus, we can associate a unique binary tree to each proposition, called the formation tree.

The formation tree contains all subformulas of a formula, starting with the formula at its root and breaking it down into its subformulas until you reach the propositional variables at its leafs.

Formation Tree

A formation tree of a proposition p has a root labeled with p and satisfies the following rules:

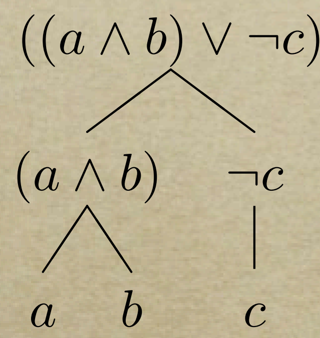
T1. Each leaf is an occurrence of a propositional variable in p.

T2. Each internal node with a single successor is labeled by a subformula ¬q of p and has q as a successor.

T3. Each internal node with two successors is labeled by a subformula aXb of p with X in $\{\wedge, \vee, \oplus, \rightarrow, \leftrightarrow\}$ and has a as a left successor and b as a right successor.

Example

Example 4. The formation tree of the formula $((a \wedge b) \vee \neg c)$ is given by



Assigning Meanings to Formulas

We know that each formula corresponds to a unique binary tree.

We can evaluate the formula by

- giving each propositional variable an interpretation.

- defining the meaning of each logical connective

- propagate the truth values from the leafs to the root in a unique way, so that we get a unambiguous evaluation of each formula.

Semantics

Let B={t,f}. Assign to each connective a function M: B->B

that determines its semantics.

P	$M_\neg(P)$
f	t
t	f

P	Q	$M_\wedge(P,Q)$	$M_\vee(P,Q)$	$M_\oplus(P,Q)$	$M_\rightarrow(P,Q)$	$M_\leftrightarrow(P,Q)$
f	f	f	f	f	t	t
f	t	f	t	t	t	f
t	f	f	t	t	f	f
t	t	t	t	f	t	t

Semantics

The semantics of the language **Prop** is given by assigning truth values to each proposition in **Prop**. Clearly, an arbitrary assignment of truth values is not interesting, since we would like everything to be consistent with the meaning of the connectives that we have just learned. For example, if the propositions a and b have been assigned the value $\mathbf{t}$, then it is reasonable to insist that $a \wedge b$ be assigned the value $\mathbf{t}$ as well. Therefore, we will introduce the concept of a valuation, which models the semantics of **Prop** in an appropriate way.

Valuations

A **valuation** $v\colon \mathbf{Prop} \to \mathbf{B}$ is a function that assigns a truth value to each proposition in **Prop** such that

V1. $v[\![\neg a]\!] = M_\neg(v[\![a]\!])$
V2. $v[\![(a \wedge b)]\!] = M_\wedge(v[\![a]\!], v[\![b]\!])$
V3. $v[\![(a \vee b)]\!] = M_\vee(v[\![a]\!], v[\![b]\!])$
V4. $v[\![(a \oplus b)]\!] = M_\oplus(v[\![a]\!], v[\![b]\!])$
V5. $v[\![(a \rightarrow b)]\!] = M_\rightarrow(v[\![a]\!], v[\![b]\!])$
V6. $v[\![(a \leftrightarrow b)]\!] = M_\leftrightarrow(v[\![a]\!], v[\![b]\!])$

holds for all propositions a and b in **Prop**. The properties **V1–V6** ensure that the valuation respects the meaning of the connectives. We can restrict a valuation v to a subset of the set of proposition. If A and B are subsets of **Prop** such that $A \subseteq B$, and $v_A\colon A \to \mathbf{B}$ and $v_B\colon B \to \mathbf{B}$ are valuations, then v_B is called an **extension** of the valuation v_A if and only if v_B coincides with v_A when restricted to A.

Uniqueness of Valuations

Theorem 1. *If two valuations v and v' coincide on the set* $\mathbf{S}$ *of symbols, then they coincide on the set* $\mathbf{Prop}$ *of all propositions.*

Proof. Seeking a contradiction, we assume that there exist two valuations v and v' that coincide on $\mathbf{S}$, but do not coincide on $\mathbf{Prop}$. Thus, the set

$$C = \{a \in \mathbf{Prop} \,|\, v[\![a]\!] \neq v'[\![a]\!]\}$$

of counter examples is not empty. Choose a counter example a in C of minimal length, where the length of the proposition is defined as the number of terminal symbols. Then a cannot be of the form $a = \neg b$, since the minimality of the counter example implies that $v[\![b]\!] = v'[\![b]\!]$, which implies

$$v[\![a]\!] = M_\neg(v[\![b]\!]) = M_\neg(v'[\![b]\!]) = v'[\![a]\!]\,.$$

Similarly, a cannot be of the form bXc for some propositions b and c in $\mathbf{Prop}$ and some connective X in $\{\wedge, \vee, \oplus, \rightarrow, \leftrightarrow\}$. Indeed, by the minimality of the counter example $v[\![b]\!] = v'[\![b]\!]$ and $v[\![c]\!] = v'[\![c]\!]$, which implies

$$v[\![a]\!] = M_X(v[\![b]\!]\,, v[\![c]\!]) = M_X(v'[\![b]\!]\,, v'[\![c]\!]) = v'[\![a]\!]\,.$$

Therefore, a must be a symbol in $\mathbf{S}$, but both valuations coincide on the set $\mathbf{S}$ of symbols, so a cannot be an element of C, which is a contradiction. $\square$

Interpretation

An **interpretation** of a proposition p in **Prop** is an assignment of truth values to all variables that occur in p. More generally, an interpretation of a set Y of propositions is an assignment of truth values to all variables that occur in formulas in Y. The previous theorem states that an interpretation of **Prop** has *at most* one extension to a valuation on **Prop**.

Interlude: Induction

Strong Induction

Suppose we wish to prove a certain assertion concerning nonnegative integers.

Let A(n) be the assertion concerning the integer n.

To prove it for all n >= 0, we can do the following:

1) Prove that the assertion A(0) is true.

2) Assuming that the assertions A(k) are proved for all k<n, prove that the assertion A(n) is true.

We can conclude that A(n) is true for all n>=0.

Example

Theorem: For all n >= 0, we have

$$1+2+\ldots+n = n(n+1)/2$$

Proof. We prove it by strong induction. The assertion A(n) is the assertion of the theorem.

For n=0, we have 0 = 0(0+1)/2, hence A(0) is true.

Suppose that the assertion A(k) is true for integers 0<=k<n.

Then $1 + 2 + \ldots + n-1 + n = (n-1)n/2 + n = ((n-1)n + 2n)/2 = (n^2+n)/2$

$= (n+1)n/2$. Therefore, A(n) is true.

By the principle of strong induction, A(n) is true for all n>=0.

End of Interlude

Degree

It remains to show that each interpretation of **Prop** has an extension to a valuation. For this purpose, we define the **degree** of a proposition p in **Prop**, denote $\deg p$, as the number of occurrences of logical connectives in p. In other words, the degree function satisfies the following properties:

D1. An element in **S** has degree 0.
D2. If a in **Prop** has degree n, then $\neg a$ has degree $n + 1$.
D3. If a and b in **Prop** are respectively of degree n_a and n_b, then aXb is of degree $n_a + n_b + 1$ for all connectives X in $\{\wedge, \vee, \oplus, \rightarrow, \leftrightarrow\}$.

Example 5. The proposition $((a \wedge b) \vee \neg c)$ is of degree 3.

Extensions of Interpretations

Theorem 2. *Each interpretation of* ***Prop*** *has a unique extension to a valuation.*

Proof. We will show by induction on the degree of a proposition that an interpretation $v_0 \colon \mathbf{S} \to \mathbf{B}$ has an extension to a valuation $v \colon \mathbf{Prop} \to \mathbf{B}$. The uniqueness of this extension is obvious from Theorem 1.

We set $v(a) = v_0(a)$ for all a of degree 0. Then v is certainly a valuation on the set of degree 0 propositions.

Suppose that v is a valuation for all propositions of degree less than n extending v_0. If a is a proposition of degree n, then it has a unique formation tree. The immediate successors of a in the formation tree are labeled by subformulas of a of degree less than n; hence, these successors have a valuation assigned. Therefore, v has a unique extension to a using the consistency rules **V1**–**V6**. We can conclude that v is a valuation on the set of all proposition of degree n extending v_0. Therefore, the claim follows by induction. $\square$

Summary

Summary. Informally, we can summarize the meaning of the connectives as follows:

1) The and connective $(a \wedge b)$ is true if and only if both a and b are true.
2) The or connective $(a \vee b)$ is true if and only if at least one of a, b is true.
3) The exclusive or $(a \oplus b)$ is true if and only if precisely one of a, b is true.
4) The implication $(a \rightarrow b)$ is false if and only if the premise a is true and the conclusion b is false.
5) The biconditional connective $(a \leftrightarrow b)$ is true if and only if the truth values of a and b are the same.

An interpretation of a subset S of **Prop** is an assignment of truth values to all variables that occur in the propositions contained in S. We showed that there exist a unique valuation extending an interpretation of all propositions.