Proof by Induction

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Motivation

Induction is an axiom which allows us to prove that certain properties are true for all positive integers (or for all nonnegative integers, or all integers >= some fixed number)

Induction Principle

Let $A(n)$ be an assertion concerning the integer n.

If we want to show that $A(n)$ holds for all positive integer n, we can proceed as follows:

Induction basis: Show that the assertion $A(1)$ holds.

Induction step: For all positive integers n, show that $A(n)$ implies $A(n+1)$.

Standard Example

For all positive integers n, we have

$A(n) = 1+2+...+n = n(n+1)/2$

Induction basis:

Since $1 = 1(1+1)/2$, the assertion $A(1)$ is true.

Induction step:

Suppose that $A(n)$ holds. Then

$1+2+...+n+(n+1) = n(n+1)/2 + n+1 = (n^2 + n+2n+2)/2$
$= (n+1)(n+2)/2$,

hence $A(n+1)$ holds. Therefore, the claim follows by induction on n.

The Main Points

We established in the induction basis that the assertion $A(1)$ is true.

We showed in the induction step that $A(n+1)$ holds, assuming that $A(n)$ holds.

In other words, we showed in the induction step that $A(n) \to A(n+1)$ holds for all $n \geq 1$.

Example 2

Theorem: For all positive integers n, we have

$$1+3+5+...+(2n-1) = n^2$$

Proof. We prove this by induction on n. Let $A(n)$ be the assertion of the theorem.

Induction basis: Since $1 = 1^2$, it follows that $A(1)$ holds.

Induction step: Suppose that $A(n)$ holds. Then

$$1+3+5+...+(2n-1)+(2n+1) = n^2+2n+1 = (n+1)^2$$

holds. In other words, $A(n)$ implies $A(n+1)$.

Quiz

Theorem: We have

$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$

for all n >= 1.

Proof. Your turn!!!

Let B(n) denote the assertion of the theorem.

Induction basis:

Since $1^2 = 1(1+1)(2+1)/6$, we can conclude that B(1) holds.

Quiz

Inductive step: Suppose that B(n) holds. Then

$1^2 + 2^2 + ... + n^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$

Expanding the right hand side yields

$n^3/3 + 3n^2/2 + 13n/6 + 1$

One easily verifies that this is equal to

$(n+1)(n+2)(2(n+1)+1)/6$

Thus, B(n+1) holds.

Therefore, the proof follows by induction on n.

Tip

How can you verify whether your algebra is correct?

Use http://www.wolframalpha.com

[Not allowed in any exams, though. Sorry!]

What's Wrong?

Billiard Balls

"**Theorem**": All billiard balls have the same color.

Proof: By induction, on the number of billiard balls.

Induction basis:

Our theorem is certainly true for n=1.

What's wrong?

Induction step:

Assume the theorem holds for n billiard balls. We prove it for n+1. Look at the first n billiard balls among the n+1. By induction hypothesis, they have the same color. Now look at the last n billiard balls. They have the same color. Hence all n+1 billiard balls have the same color.

Weird Properties of Positive Integers

"**Theorem**": For all positive integers n, we have n=n+1.

"Proof": Suppose that the claim is true for n=k. Then

k+1 = (k) + 1 = (k+1) + 1

by induction hypothesis. Thus, k+1=k+2.

Therefore, the theorem follows by induction on n.

What's wrong?

Maximally Weird!

"**Theorem**": For all positive integers n, if a and b are positive integers such that max{a,b}=n, then a=b.

Proof: By induction on n. The result holds for n = 1, i.e., if max {a, b} = 1, then a = b = 1.

Suppose it holds for n, i.e., if max {a,b} = n, then a = b. Now suppose max {a, b} = n + 1.
Case 1: a – 1 ≥ b – 1. Then a≥b. Hence a=max{a,b}=n+1.
Thus a – 1 = n and max {a – 1, b – 1} = n.
By induction, a–1=b–1. Hence a=b.
Case 2: b – 1 ≥ a – 1.
Same argument.

Maximally Weird!!

Fallacy: In the proof we used the inductive hypothesis to conclude $\max\{a - 1, b - 1\} = n \Rightarrow a - 1 = b - 1$.

However, we can only use the inductive hypothesis if $a - 1$ and $b - 1$ are positive integers. This does not have to be the case as the example $b=1$ shows.

More Examples

Factorials

Theorem. $\sum_{i=0}^{n} i(i!) = (n+1)! - 1.$

By convention: $0! = 1$
Induction basis:
Since $0 = 1 - 1$, the claim holds for $n = 0$.
Induction Step:
Suppose the claim is true for n. Then

$$\begin{aligned}
\sum_{i=0}^{n+1} i(i!) &= (n+1)(n+1)! + \sum_{i=0}^{n} i(i!) \\
&= (n+1)(n+1)! + (n+1)! - 1 \text{ by ind. hyp.} \\
&= (n+2)(n+1)! - 1 \\
&= (n+2)! - 1
\end{aligned}$$

Divisibility

Theorem: For all positive integers n, the number

$7^n - 2^n$

is divisible by 5.

Proof: By induction.

Induction basis. Since 7-2=5, the theorem holds for n=1.

Divisibility

Inductive step:

Suppose that 7^n-2^n is divisible by 5. Our goal is to show that this implies that $7^{n+1}-2^{n+1}$ is divisible by 5. We note that

$7^{n+1}-2^{n+1} = 7x7^n-2x2^n= 5x7^n+2x7^n-2x2^n = 5x7^n +2(7^n-2^n)$.

By induction hypothesis, $(7^n-2^n) = 5k$ for some integer k.

Hence, $7^{n+1}-2^{n+1}= 5x7^n +2x5k = 5(7^n +2k)$, so

$7^{n+1}-2^{n+1} =5$ x some integer.

Thus, the claim follows by induction on n.

Strong Induction

Strong Induction

Suppose we wish to prove a certain assertion concerning positive integers.

Let A(n) be the assertion concerning the integer n.

To prove it for all n >= 1, we can do the following:

1) Prove that the assertion A(1) is true.

2) Assuming that the assertions A(k) are proved for all $k<n$, prove that the assertion A(n) is true.

We can conclude that A(n) is true for all n>=1.

Strong Induction

Induction basis:

Show that $A(1)$ is true.

Induction step:

Show that $(A(1) \wedge \ldots \wedge A(n)) \rightarrow A(n+1)$

holds for all n >=1.

Postage

Theorem: Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

Postage

Proof by induction on the amount of postage.

Induction Basis:

If the postage is

12 cents = use three 4 cent stamps

13 cents = use two 4-cent and one 5-cent stamp.

14 cents = use one 4-cent and two 5-cent stamps.

15 cents = use three 5-cent stamps.

Postage

Inductive step:

Suppose that we have shown how to construct postage for every value from 12 up through k. We need to show how to construct k + 1 cents of postage.

Since we've already proved the induction basis, we may assume that $k + 1 \geq 16$. Since $k+1 \geq 16$, we have $(k+1)-4 \geq 12$. By inductive hypothesis, we can construct postage for $(k + 1) - 4$ cents using m 4-cent stamps and n 5-cent stamps for some non-negative integers m and n. In other words $((k + 1) - 4) = 4m + 5n$; hence, $k+1 = 4(m+1)+5n$.

Quiz

Why did we need to establish four cases in the induction basis?

Isn't it enough to remark that the postage for 12 cents is given by three 4 cents stamps?

Another Example: Sequence

Theorem: Let a sequence (a_n) be defined as follows:

$a_0=1$, $a_1=2$, $a_2=3$,

$a_k = a_{k-1}+a_{k-2}+a_{k-3}$ for all integers $k\geq3$.

Then $a_n \leq 2^n$ for all integers $n\geq0$. P(n)

Proof. Induction basis:

The statement is true for $n=0$, since $a_0=1 \leq 1=2^0$ P(0)

for $n=1$: since $a_1=2 \leq 2=2^1$ P(1)

for $n=2$: since $a_2=3 \leq 4=2^2$ P(2)

Sequence (cont'd)

Inductive step:

Assume that P(i) is true for all i with $0 \le i < k$, that is,

$a_i \le 2^i$ for all $0 \le i < k$, where $k > 2$.

Show that P(k) is true: $a_k \le 2^k$

$a_k = a_{k-1} + a_{k-2} + a_{k-3} \le 2^{k-1} + 2^{k-2} + 2^{k-3}$

$\le 2^0 + 2^1 + \ldots + 2^{k-3} + 2^{k-2} + 2^{k-1}$

$= 2^k - 1 \le 2^k$

Thus, P(n) is true by strong induction.