Problem Set 1
CPSC 411 Analysis of Algorithms
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The assignment is due next Monday, Feb 2, 2009, before class.

The limit and limit superior criteria give in Lecture 2 for determining whether $f(n) = O(g(n))$ can be convenient in some of the subsequent exercises.

Exercise 1. *Suppose that $p(n) = a_0 + a_1 n + \cdots + a_m n^m$ is a polynomial of degree m with complex coefficients. Show that $p(n) = O(n^k)$ for all $k \geq m$, and $p(n) \neq O(n^\ell)$ for $0 \leq \ell < m$.*

Exercise 2. *Show that for fixed k, we have*

$$\binom{n}{k} = \frac{n^k}{k!} + O(n^{k-1}) \quad \textit{and} \quad \binom{n+k}{k} = \frac{n^k}{k!} + O(n^{k-1}),$$

where $\binom{n}{k} = n(n-1)(n-2)\cdots(n-k+1)/k!$ is the binomial coefficient.

Exercise 3. *Show that $n/(n+1) = 1 + O(1/n)$.*

Exercise 4. *Prove or disprove: $e^n = O(2^n)$.*

Remark. Let n_0 and n_1 be positive integers with $n_1 > n_0$. If $L(n_0) \geq R(n_0)$ holds, and the numbers $L(n), R(n), L(n+1)$, and $R(n+1)$ are positive and

$$\frac{L(n+1)}{L(n)} \geq \frac{R(n+1)}{R(n)}$$

holds for all n in the range $n_0 \leq n < n_1$, then it follows that $L(n) \geq R(n)$ holds for all n in the range $n_0 \leq n \leq n_1$.

Exercise 5. *(a) Prove by induction on k that*

$$\left(1 + \frac{1}{n}\right)^k < 1 + \frac{k}{n} + \frac{k^2}{n^2}$$

holds for all k in $\{1, 2 \ldots, n\}$.
(b) Prove by induction that $n! \geq (n/3)^n$ holds for all $n \geq 1$.
[Hint: Use part (a) of this exercise and the above remark].
(c) Deduce that $\log(n!) = \Omega(n \log n)$ holds.

Exercise 6 (CLRS)**.** *Exercise 2.3-3, page 36*