FRAMES IN QUANTUM AND CLASSICAL INFORMATION THEORY

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A FRAME

- A sequence $\{x_i\}$ of vectors in a Hilbert space with the property that there are constants $A, B \geq 0$ such that

$$A\|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B\|x\|^2$$

for all x in the Hilbert space.

- Examples?

A SOURCE OF INFORMATION

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- Example: coin tossing with $\mathcal{X} = \{H, T\}$.

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- Example: $\mathcal{X} = \{0, 1, 2, 3\}$,

$$|\psi_0\rangle = \alpha_0|e_0\rangle + \beta_0|e_1\rangle \quad |\psi_1\rangle = \alpha_1|e_0\rangle + \beta_1|e_1\rangle$$
$$|\psi_2\rangle = \alpha_2|e_0\rangle + \beta_2|e_1\rangle \quad |\psi_3\rangle = \alpha_3|e_0\rangle + \beta_3|e_1\rangle.$$

QUANTUM DISCRETE MEMORYLESS SOURCE

The Density Matrix and Von Neumann Entropy

- Source density matrix:

$$\rho = \sum_{a \in \mathcal{X}} P_a |\psi_a\rangle\langle\psi_a|.$$

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- Von Neumann entropy of the source:

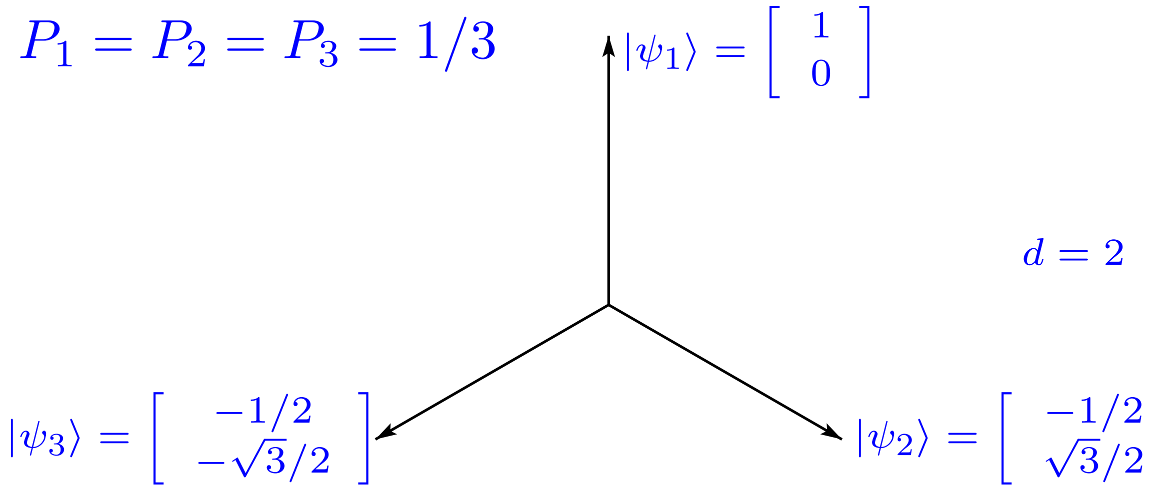
$$\begin{aligned} S(\rho) &= -\operatorname{Tr} \rho \log \rho \\ &= -\sum_i \lambda_i \log \lambda_i, \end{aligned}$$

where λ_i are the eigenvalues of ρ.

QUANTUM DISCRETE MEMORYLESS SOURCE

MB Example

$$\mathcal{X} = \{1, 2, 3\} \quad P_1 = P_2 = P_3 = 1/3$$



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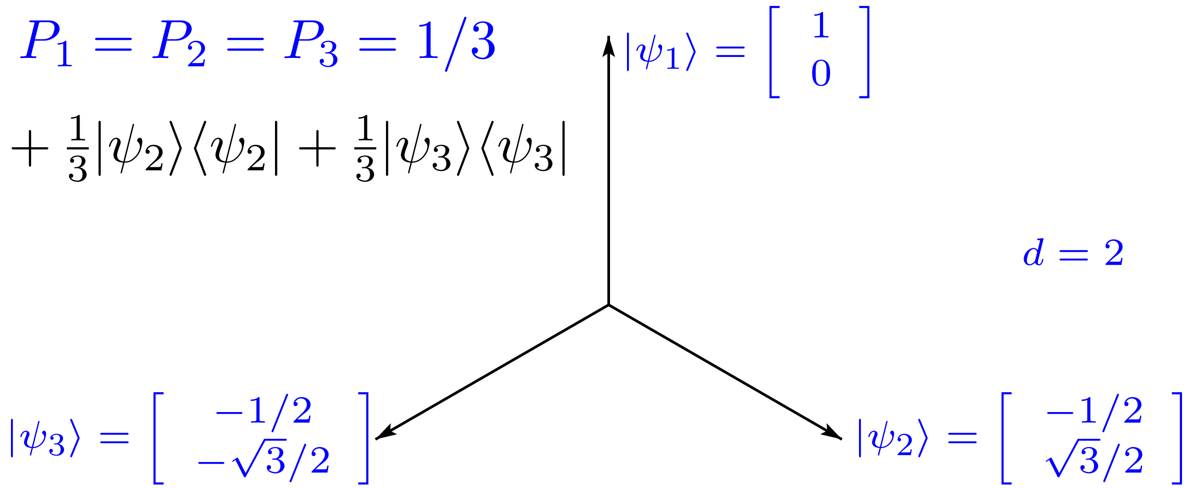

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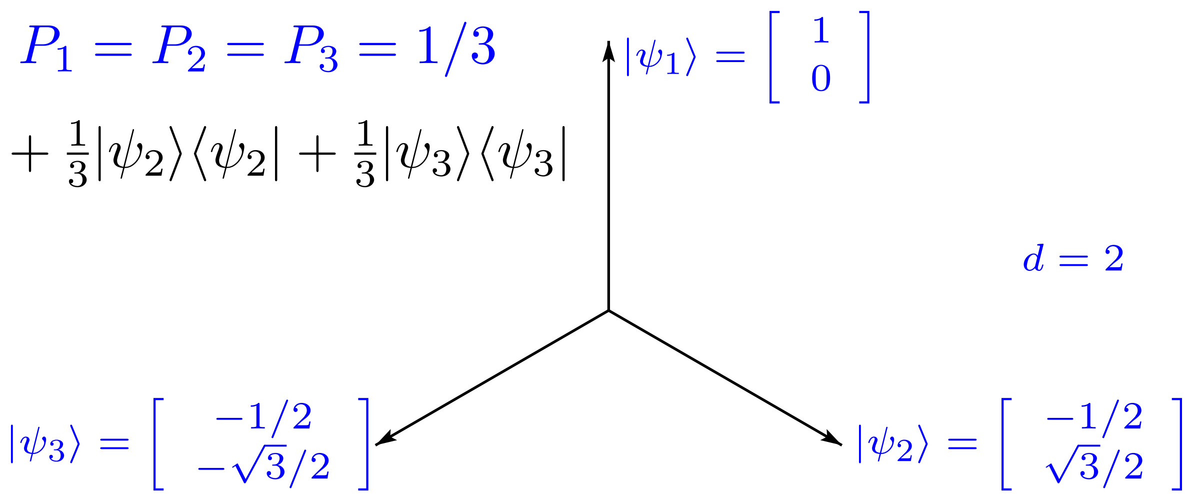
$$S(\rho) = 1$$



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Vector Sequences

- Sequences of length n are d^n-dimensional vectors.
- Source vector-sequence (state):

$$|\Psi_{\boldsymbol{x}}\rangle = |\psi_{x_1}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle, \qquad x_i \in \mathcal{X}.$$

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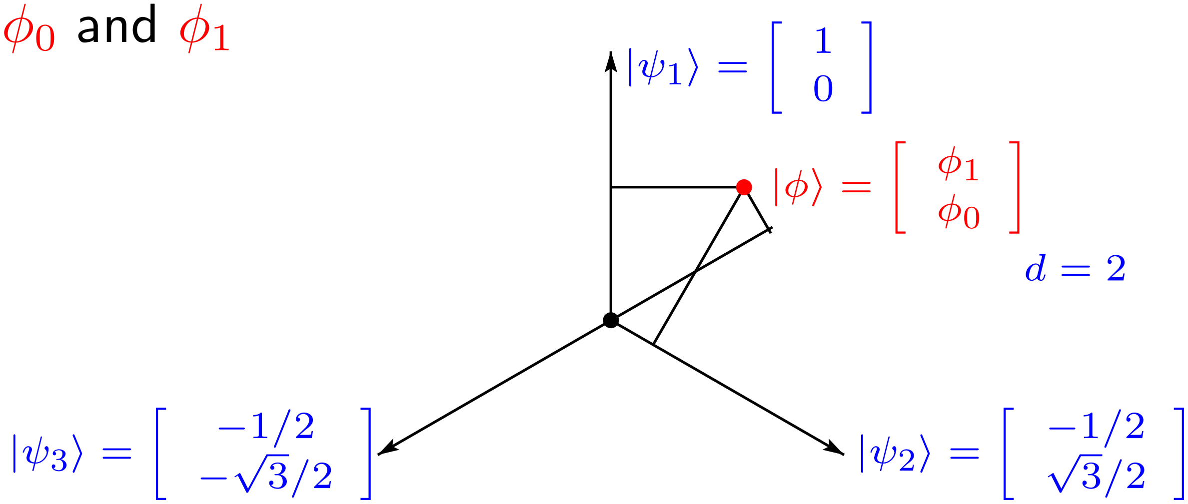
- Among all states that come from the source, we can distinguish

$$2^{n(S(\rho)-\varepsilon_n)}$$

reliably.

SENDING PACKETS OVER LOSSY NETWORKS
MB Example

Send $|\phi\rangle$ by sending ϕ_0 and ϕ_1



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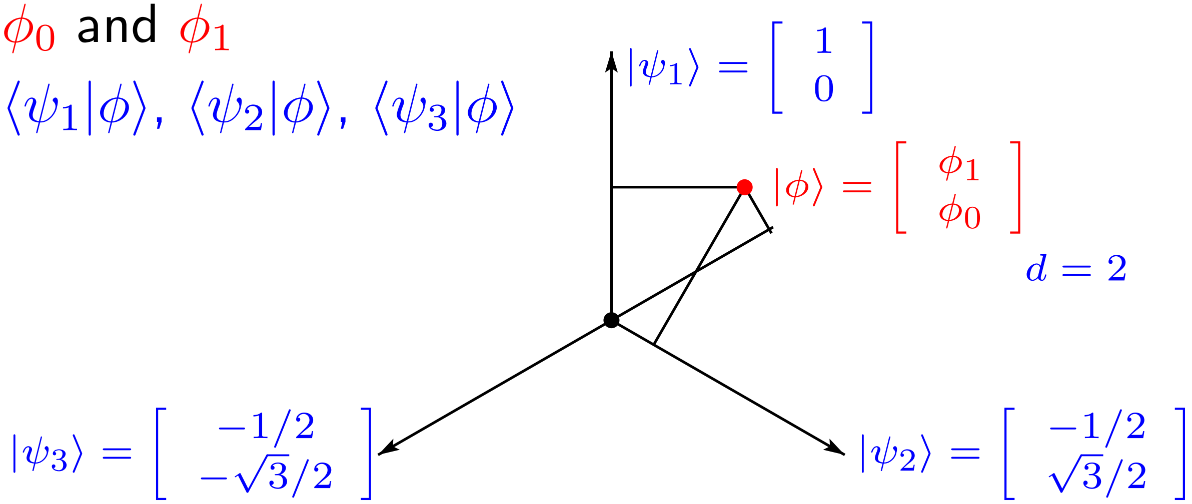

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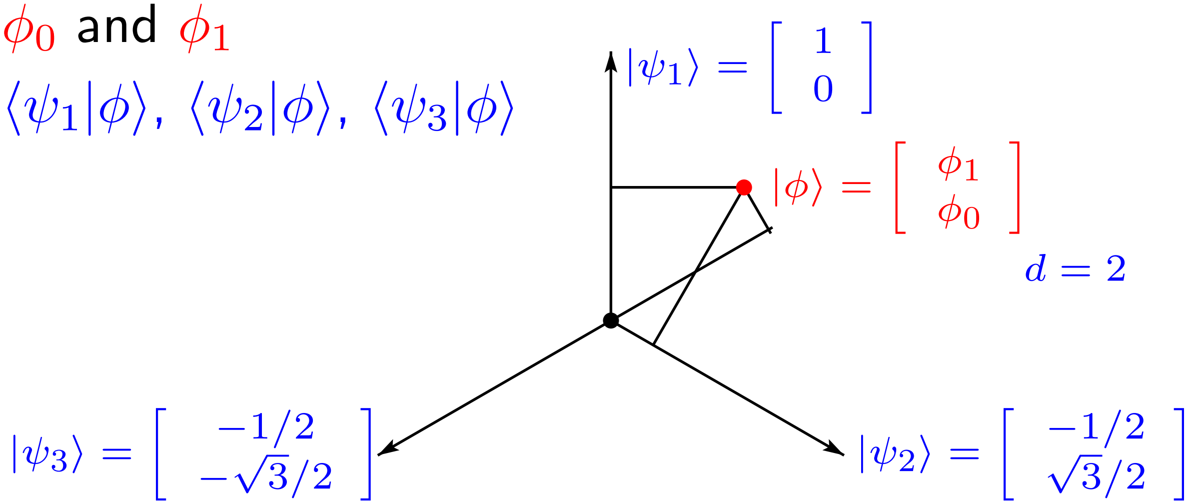

$$|\phi\rangle = \frac{2}{3}(\langle\psi_1|\phi\rangle|\psi_1\rangle + \langle\psi_2|\phi\rangle|\psi_2\rangle + \langle\psi_3|\phi\rangle|\psi_3\rangle)$$

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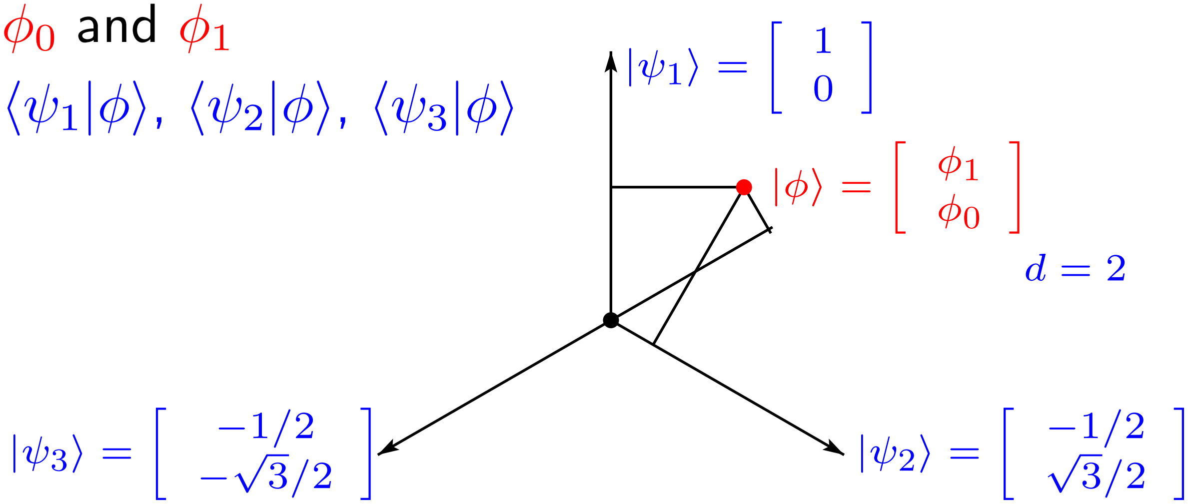

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SYNCHRONOUS CDMA SYSTEMS

K users and processing gain N

- Each user has a signature $N \times 1$ length-$\sqrt{N}$ complex vector.
- Let $\boldsymbol{s_i}$ be the signature and p_i the power of user i for $1 \leq i \leq K$.

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- Let $\boldsymbol{s_i}$ be the signature and p_i the power of user i for $1 \leq i \leq K$.
- The received vector is given by

$$\boldsymbol{r} = \sum_{i=1}^{K} \sqrt{p_i} b_i \boldsymbol{s_i} + \boldsymbol{n}$$

where

– b_i is the information symbol, for user i, $E[b_i] = 0$, $E[b_i^2] = 1$;
– $\boldsymbol{n}$ is the (Gaussian) noise vector; $E[\boldsymbol{n}] = \mathbf{0}$, $E[\boldsymbol{n}\boldsymbol{n}^\dagger] = \sigma^2 I_N$.

SYNCHRONOUS CDMA SYSTEMS

The Sum Capacity

- Let user signatures and powers be given:

$$S = [\boldsymbol{s}_{\mathbf{1}}, \ldots \boldsymbol{s}_K] \text{ and } P = \text{diag}\{p_1, \ldots, p_K\}$$

- The sum capacity:

$$C_{\text{sum}} = \frac{1}{2}\log[\det(I_N + \sigma^{-2} S P S^{\dagger})]$$

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The Object of Interest

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- Two matrices:
  - F is the $d \times K$ matrix whose columns are $\sqrt{p_i}|\psi_i\rangle$. Thus

$$FF^\dagger = \sum_{i=1}^{K} p_i |\psi_i\rangle\langle\psi_i|.$$

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  - $FF^\dagger$ is the density matrix (frame operator)
  - $F^\dagger F$ is the Gram matrix.

A FRAME

- An ensemble $\{\sqrt{p_i}|\psi_i\rangle\}$ of vectors in a Hilbert space with the property that there are constants $A, B \geq 0$ such that

$$A\langle\varphi|\varphi\rangle \leq \sum_i p_i|\langle\varphi|\psi_i\rangle|^2 \leq B\langle\varphi|\varphi\rangle$$

for all $|\varphi\rangle$ in the Hilbert space.

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- Equivalently,

$$AI_d \leq FF^\dagger \leq BI_d$$

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Information Measures

- The Von Neumann entropy:

$$S = -\operatorname{Tr} FF^\dagger \log FF^\dagger$$

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- Let $\{p_i\}$ be a PD, and $p_1 \geq p_2 \geq \cdots \geq p_K$.
- Let ρ be a density matrix, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ its eigenvalues.
- There exist vectors $|\psi_i\rangle$ such that $\rho = \sum_{i=1}^{K} p_i |\psi_i\rangle\langle\psi_i|$ iff

$$\sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n} \lambda_i \quad \text{for all } n < d.$$

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- In CDMA, user i is said to be oversized if

$$p_i > \frac{\sum_{j=i+1}^{K} p_j}{d - i}$$

INTERFERENCE MEASURE IN CDMA
Total Square Correlation (TSC)

- The Welch's lower bound to TSC (frame potential):

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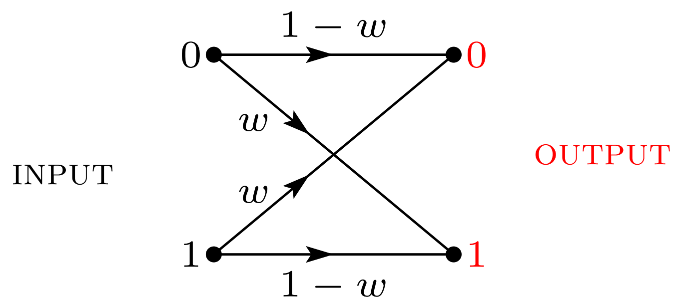
- What does reducing TSC mean?

SOME COMMUNICATION CHANNELS

- Binary Symmetric Channel:



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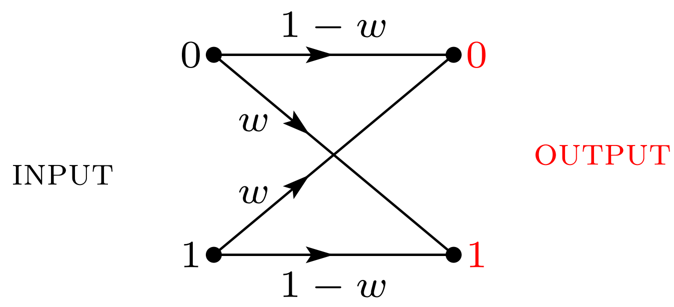
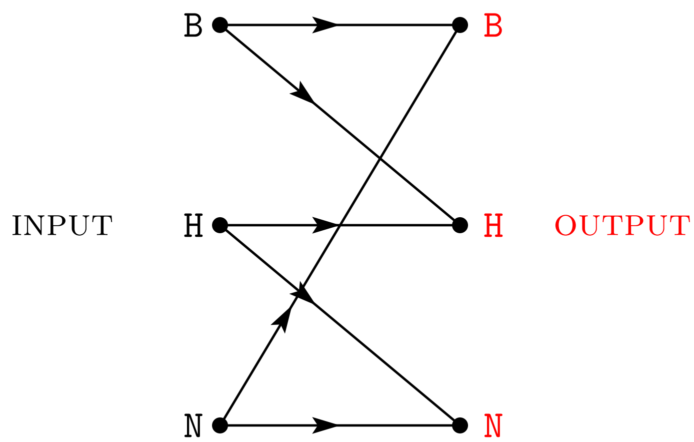
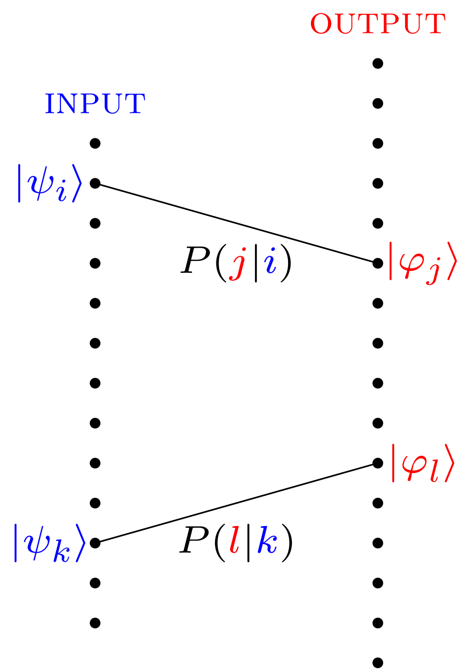


- Noisy Typewriter:



A CQ COMMUNICATION CHANNEL
A Probabilistic Device

- Inputs: vectors $|\psi_i\rangle$, $i \in \mathcal{X}$.
- Outputs: vectors $|\varphi_j\rangle$ determined by the chosen measurement.
- Transition probabilities determined by the chosen measurement.



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Von Neumann's Measurement

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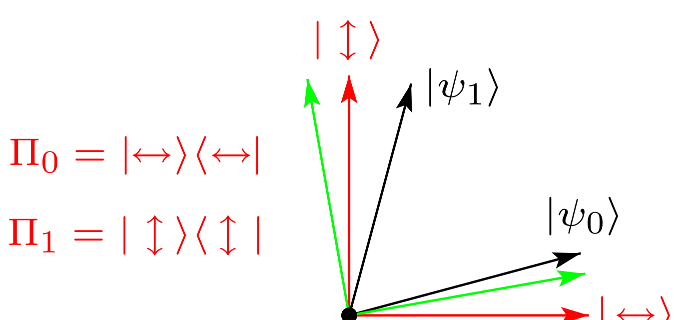

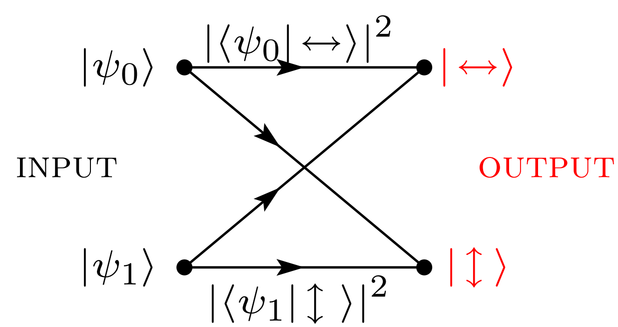

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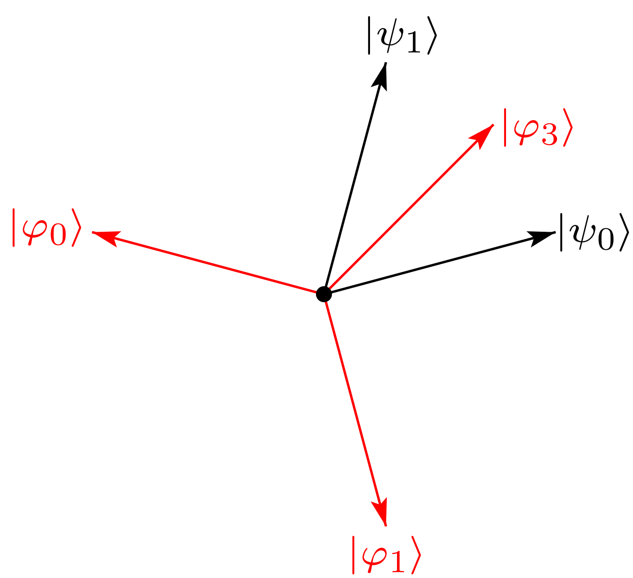

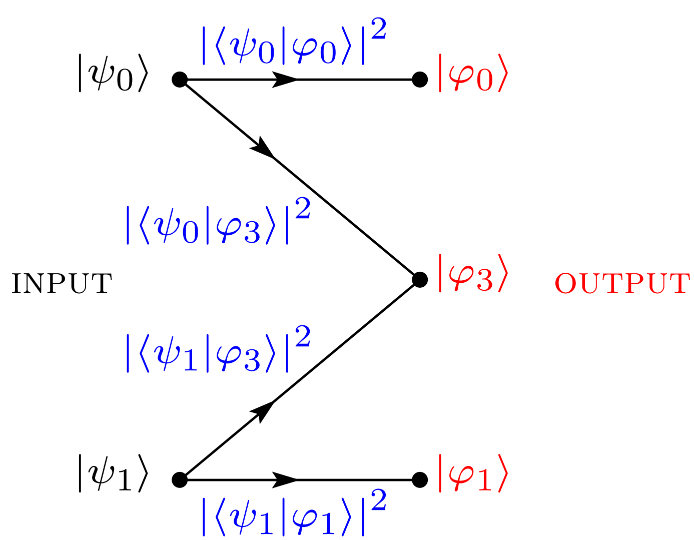

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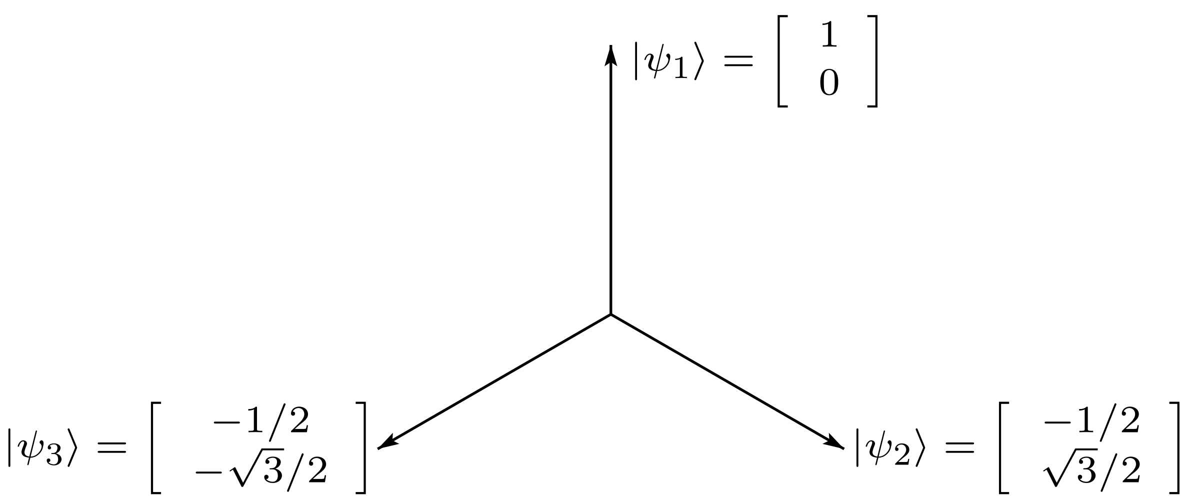

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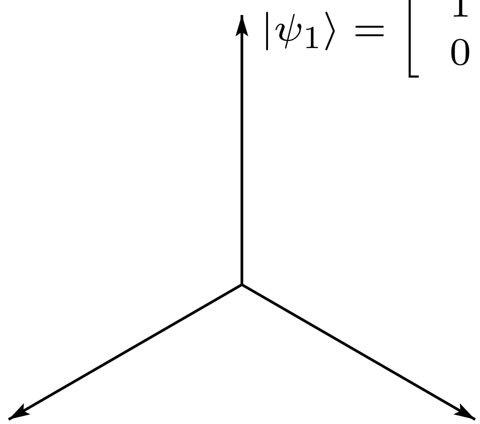

$$|\widetilde{\psi_1}\rangle = \begin{bmatrix} \sqrt{1-\alpha} \\ 0 \\ \sqrt{\alpha} \end{bmatrix}$$

$$|\psi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\widetilde{\psi_3}\rangle = \begin{bmatrix} -\sqrt{1-\alpha}/2 \\ -\sqrt{3}\sqrt{1-\alpha}/2 \\ \sqrt{\alpha} \end{bmatrix}$$

$$|\psi_3\rangle = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$|\widetilde{\psi_2}\rangle = \begin{bmatrix} -\sqrt{1-\alpha}/2 \\ \sqrt{3}\sqrt{1-\alpha}/2 \\ \sqrt{\alpha} \end{bmatrix}$$

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- Discrete: produces sequences of letters.
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- Memoryless: each letter is produced independently.
- Probability of letter a is P_a.
- Example: coin tossing with $\mathcal{X} = \{H, T\}$.

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Sequences and "Large" Sets

- Source sequence $\boldsymbol{x} = x_1, x_2, \ldots, x_n$ is in $\mathcal{X}^n$.
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- Consider all sequences $\boldsymbol{x}$ for which

$$\left|\frac{1}{n}N(a|\boldsymbol{x}) - P_a\right| \le \delta \text{ for every } a \in \mathcal{X}.$$

They form the set of typical sequences $\mathsf{T}^n_{P,\delta}$.

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- Set $\mathsf{T}^n_{P,\delta}$ is probabilistically large:

$$P^n(\mathsf{T}^n_{P,\delta}) \geq 1 - \epsilon_n.$$

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- The probability of typical sequences $\boldsymbol{x}$ is approximately $2^{-nH(P)}$:

$$2^{-n[H(P)+\epsilon'_n]} \leq P_{\boldsymbol{x}} \leq 2^{-n[H(P)-\epsilon'_n]}$$

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QUANTUM DISCRETE MEMORYLESS SOURCE

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- Typical states $|\Psi_{\boldsymbol{x}}\rangle \in \mathcal{H}^{2^n}$ correspond to typical sequences $\boldsymbol{x}$.

- There are approximately $2^{nH(P)}$ typical states.

QUANTUM DISCRETE MEMORYLESS SOURCE

Typical Subspace

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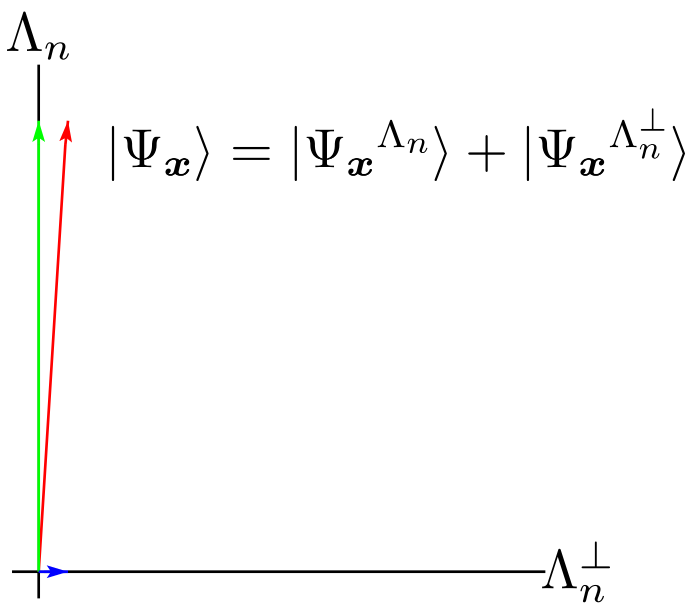

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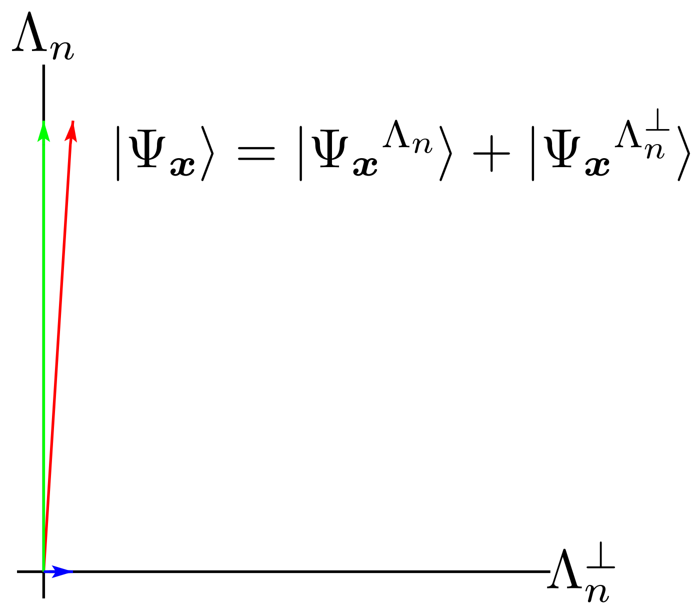

- The dimension of Λ_n is approximately $2^{nS(\rho)}$.

Code $\mathcal{C} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\} \subset \mathcal{X}^n$

- F is the $d^n \times M$ matrix whose columns are $|\psi_{\boldsymbol{x_i}}\rangle / \sqrt{M}$. Thus

$$FF^\dagger = \frac{1}{M} \sum_{i=1}^{M} |\Psi_{\boldsymbol{x}_i}\rangle\langle\Psi_{\boldsymbol{x}_i}|.$$

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ρ and σ

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- Frobenius (Hilbert-Schmidt)?

DISTANCES BETWEEN PD'S

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- Use the random coding argument!

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The Random Coding Argument

- Averaging over all codes:

$$E\Big\{\sum_{i=1}^{M}\sum_{\substack{j=1\\ j\neq i}}^{M}|\langle\Psi_{\boldsymbol{x}_i}|\Psi_{\boldsymbol{x}_j}\rangle|^2\Big\} = M(M-1)\,\mathrm{Tr}(\rho^{\otimes n}\cdot\rho^{\otimes n})$$

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$$\Big[\sum_{k=1}^{r}\big|\lambda_k-\frac{1}{r}\big|\Big]^2 \leq 2^{-n(S(\rho)-\varepsilon_n-R)}$$

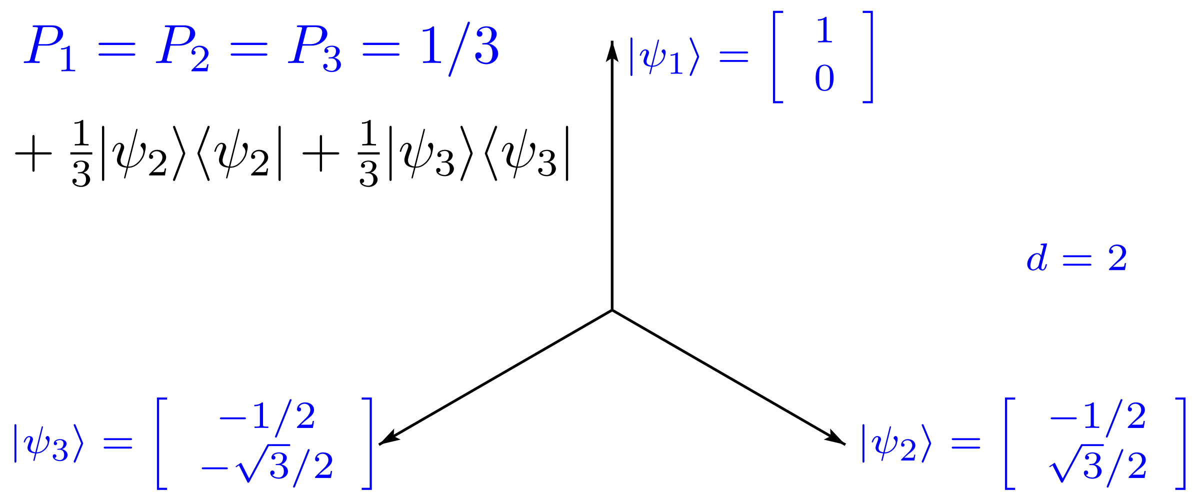
COMBINATORICS AND GEOMETRY

The MB Example

$$\mathcal{X} = \{1, 2, 3\} \quad P_1 = P_2 = P_3 = 1/3$$

$$\rho = \tfrac{1}{3}|\psi_1\rangle\langle\psi_1| + \tfrac{1}{3}|\psi_2\rangle\langle\psi_2| + \tfrac{1}{3}|\psi_3\rangle\langle\psi_3|$$
$$= \tfrac{1}{2}I$$

$$S(\rho) = 1$$



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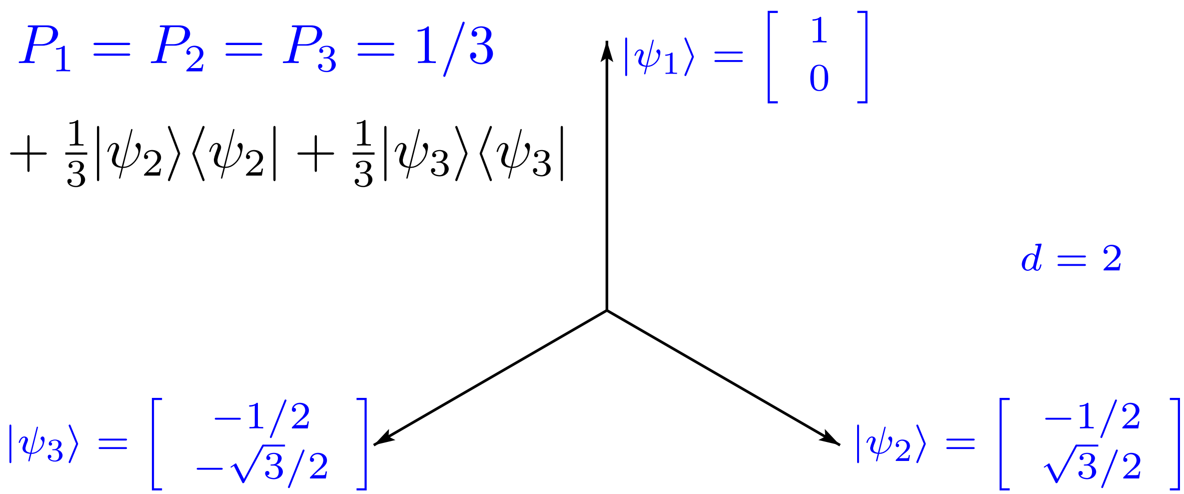

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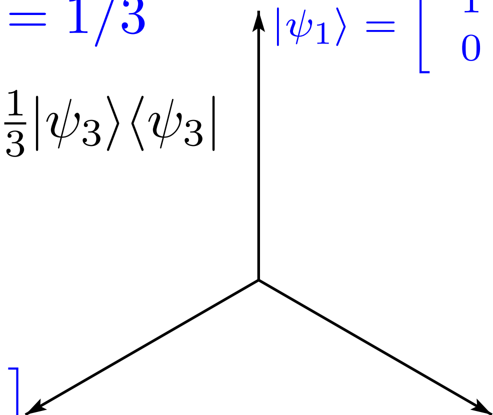

- There are 3^n typical vectors forming a frame in H^{2^n}.
- About 2^n of those vectors form a basis of H^{2^n}.
- Which ones?