CSCE 636 Neural Networks (Deep Learning)

Lecture 3: Gradient Descent and Backpropagation Algorithm

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Based on interesting lecture by Prof. Hung-yi Lee, https://www.youtube.com/watch?v=ibJpTrp5mcE

Backpropagation

Gradient Descent

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters θ^0

$$\nabla L(\theta) = \begin{bmatrix} \partial L(\theta)/\partial w_1 \\ \partial L(\theta)/\partial w_2 \\ \vdots \\ \partial L(\theta)/\partial b_1 \\ \partial L(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla L(\theta^0)$

Gradient Descent

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters $\theta^0 \longrightarrow \theta^1$

$$\nabla L(\theta) = \begin{bmatrix} \partial L(\theta)/\partial w_1 \\ \partial L(\theta)/\partial w_2 \\ \vdots \\ \partial L(\theta)/\partial b_1 \\ \partial L(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla L(\theta^0)$

$$\theta^1 = \theta^0 - \eta \nabla L(\theta^0)$$

learning rate (such as 0.001)

Gradient Descent

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters $\theta^0 \longrightarrow \theta^1 \longrightarrow \theta^2 \longrightarrow \ldots\ldots$

$$\nabla L(\theta) = \begin{bmatrix} \partial L(\theta)/\partial w_1 \\ \partial L(\theta)/\partial w_2 \\ \vdots \\ \partial L(\theta)/\partial b_1 \\ \partial L(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla L(\theta^0)$ $\quad \theta^1 = \theta^0 - \eta \nabla L(\theta^0)$

$Compute\ \nabla L(\theta^1)$ $\quad \theta^2 = \theta^1 - \eta \nabla L(\theta^1)$

Millions of parameters ……

To compute the gradients efficiently, we use ***backpropagation***.

Chain Rule

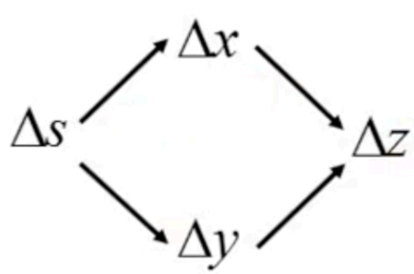
Case 1

$$y = g(x) \quad z = h(y)$$

$$\Delta x \rightarrow \Delta y \rightarrow \Delta z \qquad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

Case 2

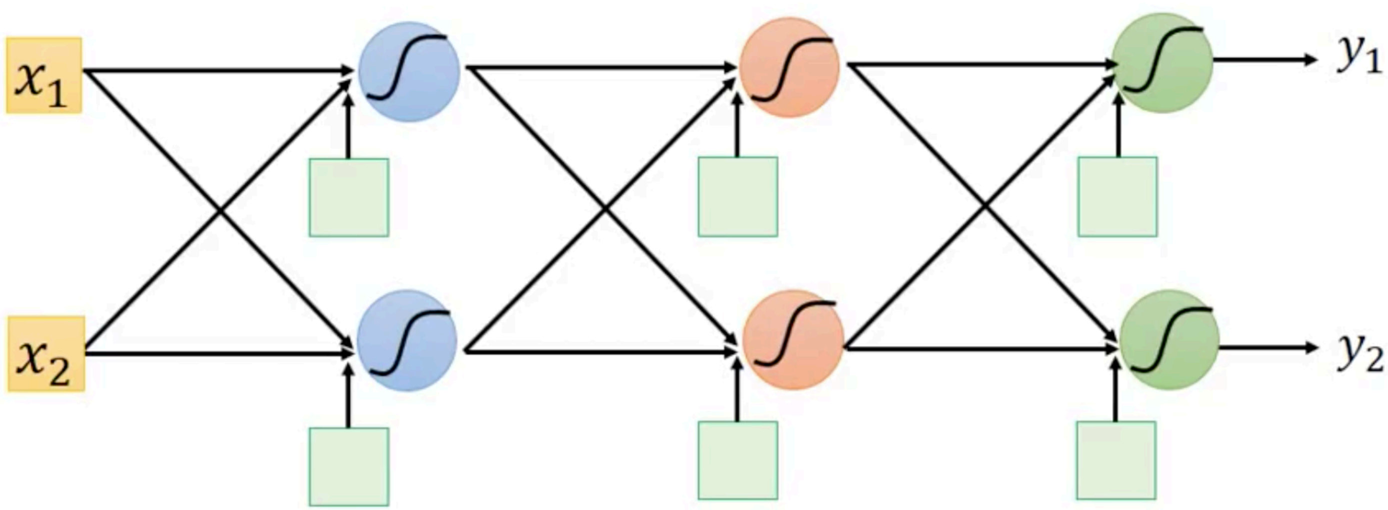
$$x = g(s) \qquad y = h(s) \qquad z = k(x, y)$$



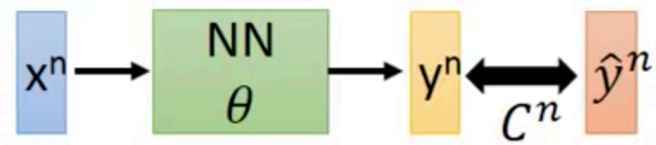
$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

Backpropagation

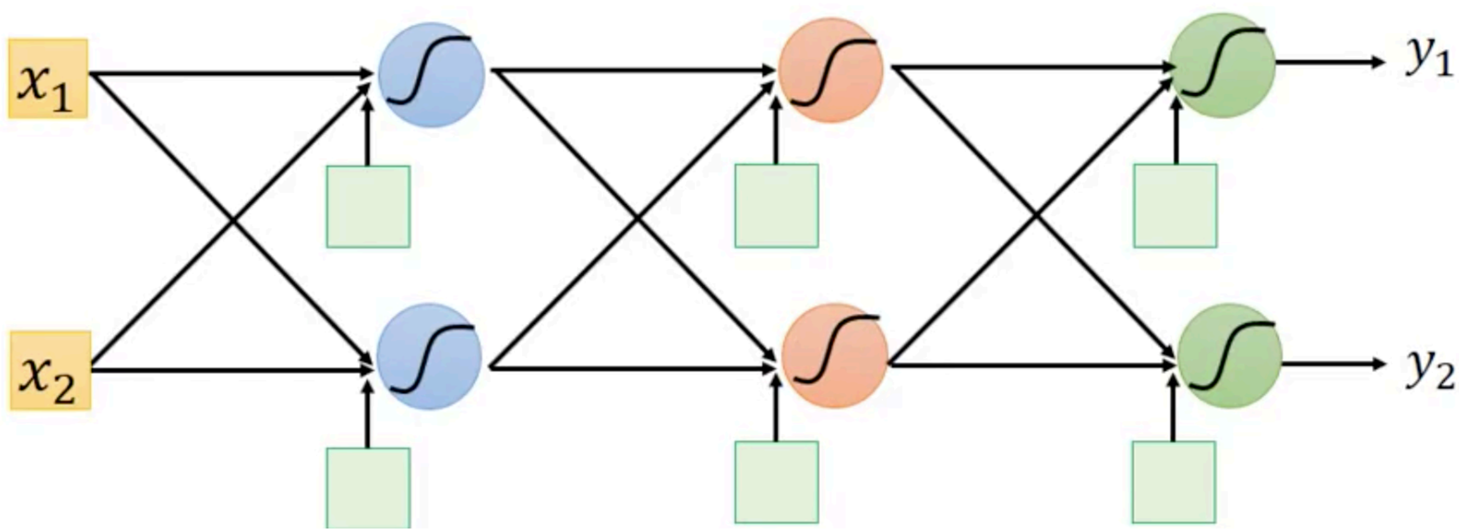
$$L(\theta) = \sum_{n=1}^{N} C^n(\theta)$$



Backpropagation

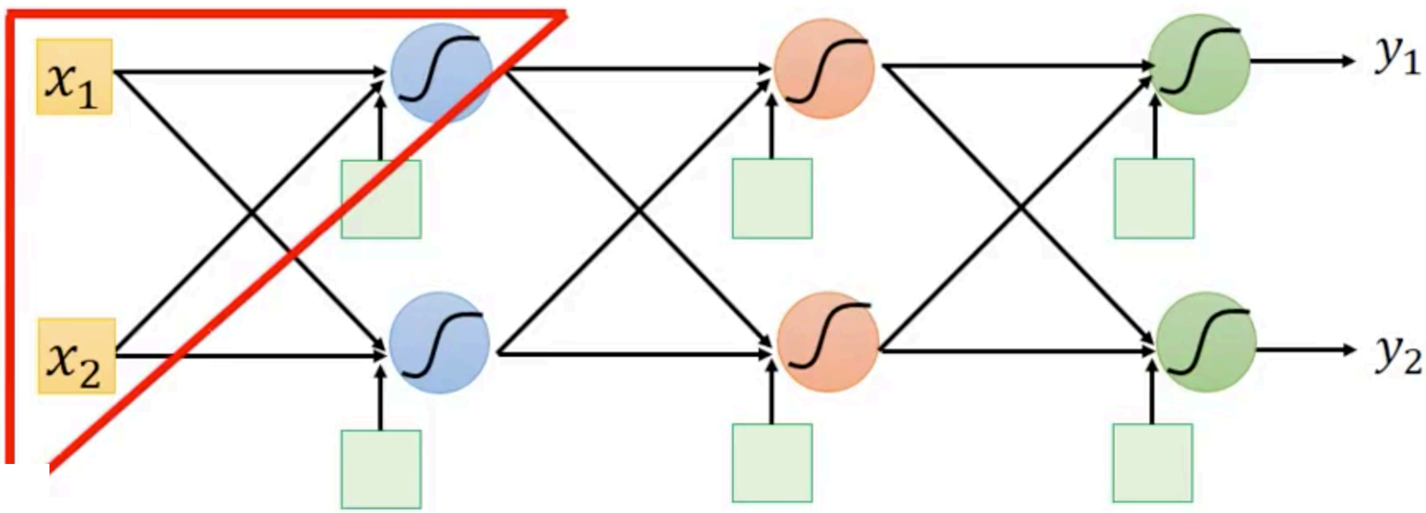


$$L(\theta) = \sum_{n=1}^{N} C^n(\theta) \quad \Rightarrow \quad \frac{\partial L(\theta)}{\partial w} = \sum_{n=1}^{N} \frac{\partial C^n(\theta)}{\partial w}$$

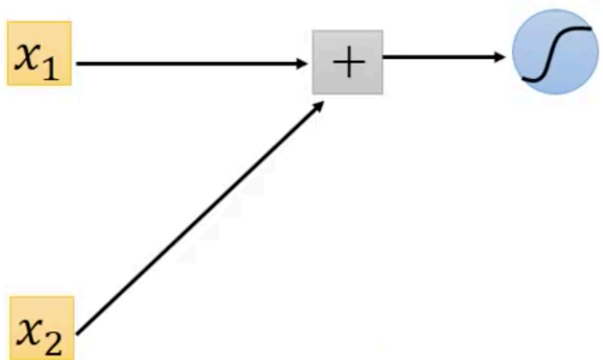


Backpropagation

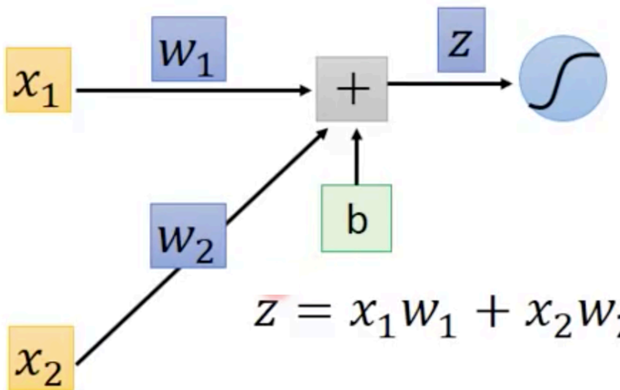
$$L(\theta) = \sum_{n=1}^{N} C^n(\theta) \quad \Rightarrow \quad \frac{\partial L(\theta)}{\partial w} = \sum_{n=1}^{N} \frac{\partial C^n(\theta)}{\partial w}$$



Backpropagation

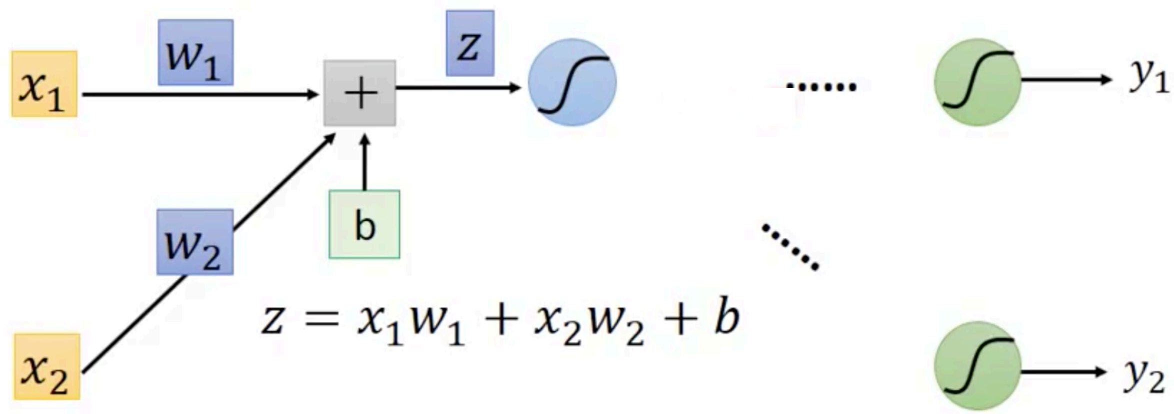


Backpropagation

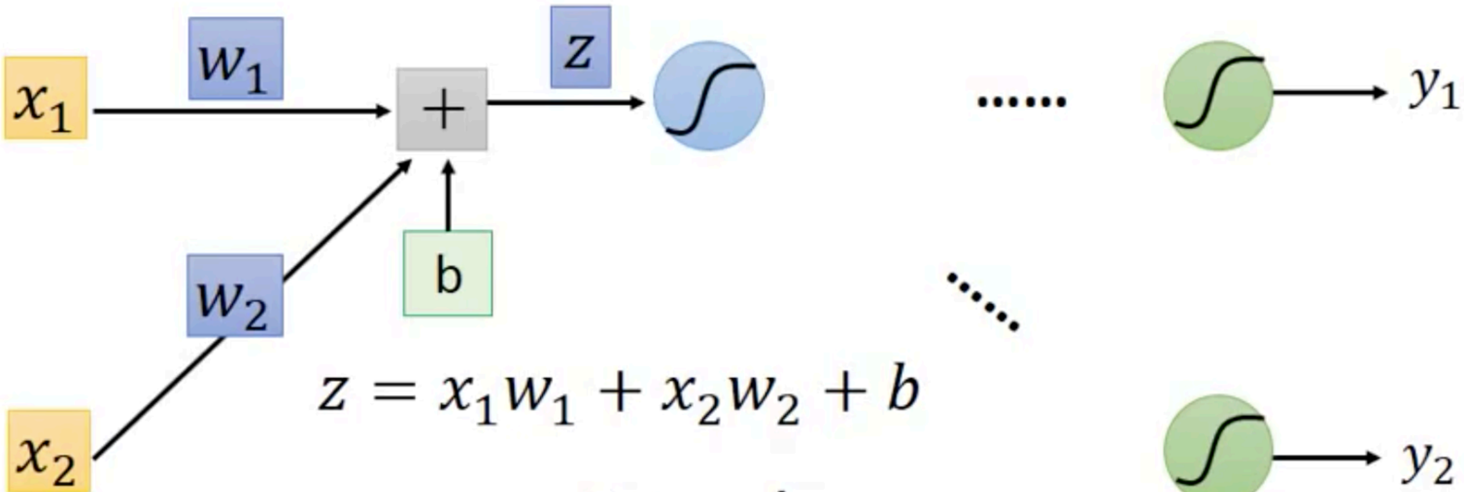


$$z = x_1 w_1 + x_2 w_2 + b$$

Backpropagation



Backpropagation



$$z = x_1 w_1 + x_2 w_2 + b$$

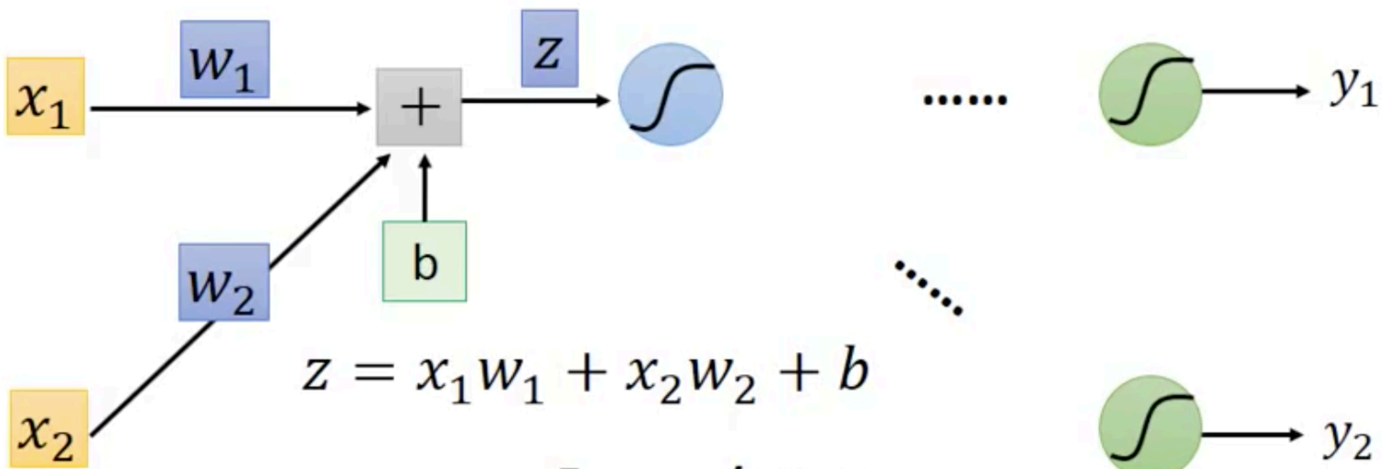
Forward pass:

Compute $\partial z / \partial w$ for all parameters

$$\frac{\partial C}{\partial w} = ? \quad \frac{\partial z}{\partial w}\frac{\partial C}{\partial z}$$

(Chain rule)

Backpropagation



$$z = x_1 w_1 + x_2 w_2 + b$$

$$\frac{\partial C}{\partial w} =? \quad \frac{\partial z}{\partial w}\frac{\partial C}{\partial z}$$

(Chain rule)

Forward pass:

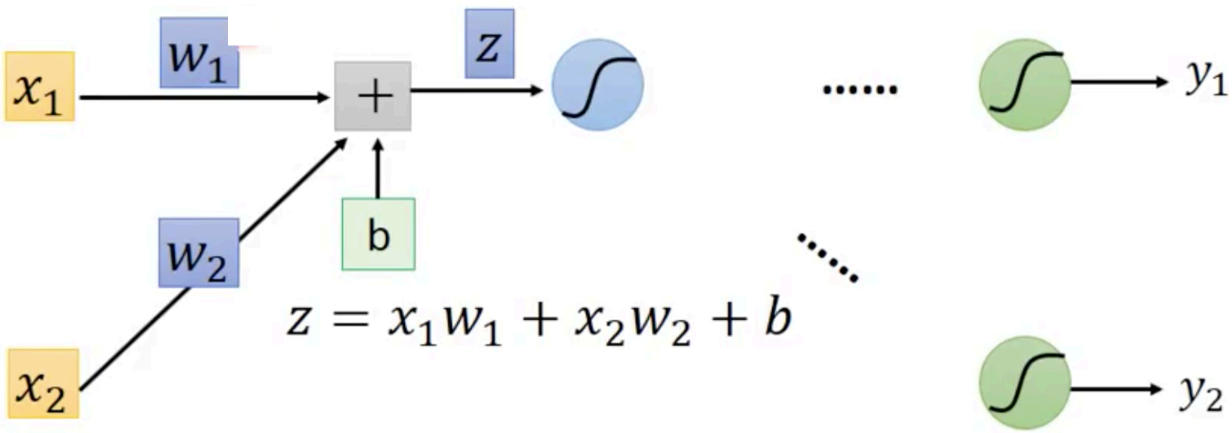
Compute $\partial z / \partial w$ for all parameters

Backward pass:

Compute $\partial C / \partial z$ for all activation function inputs z

Backpropagation – Forward pass

Compute $\partial z / \partial w$ for all parameters

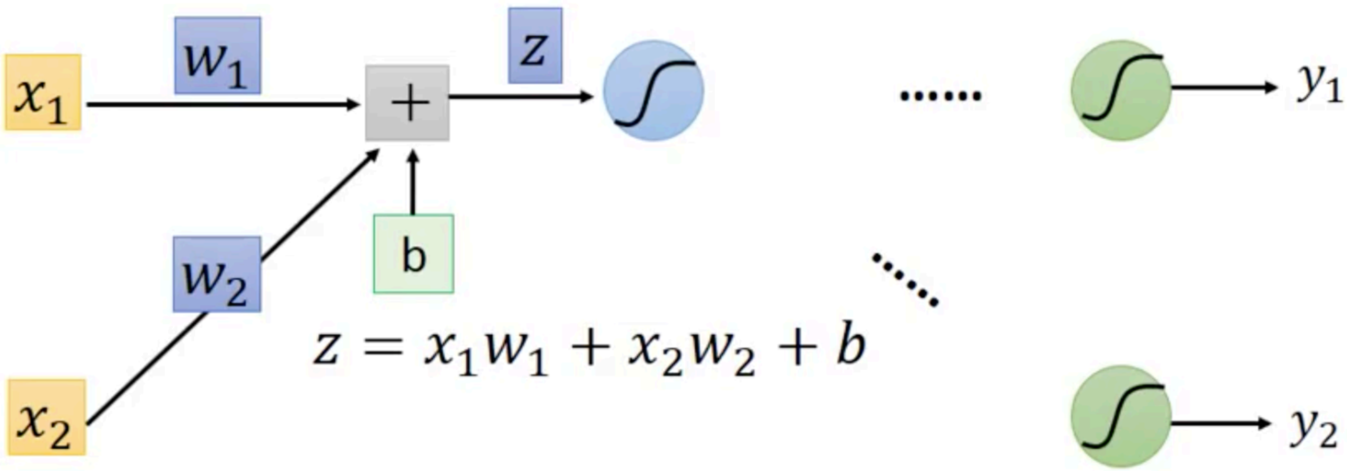


$$z = x_1 w_1 + x_2 w_2 + b$$

$\partial z / \partial w_1 = ?$

Backpropagation – Forward pass

Compute $\partial z/\partial w$ for all parameters



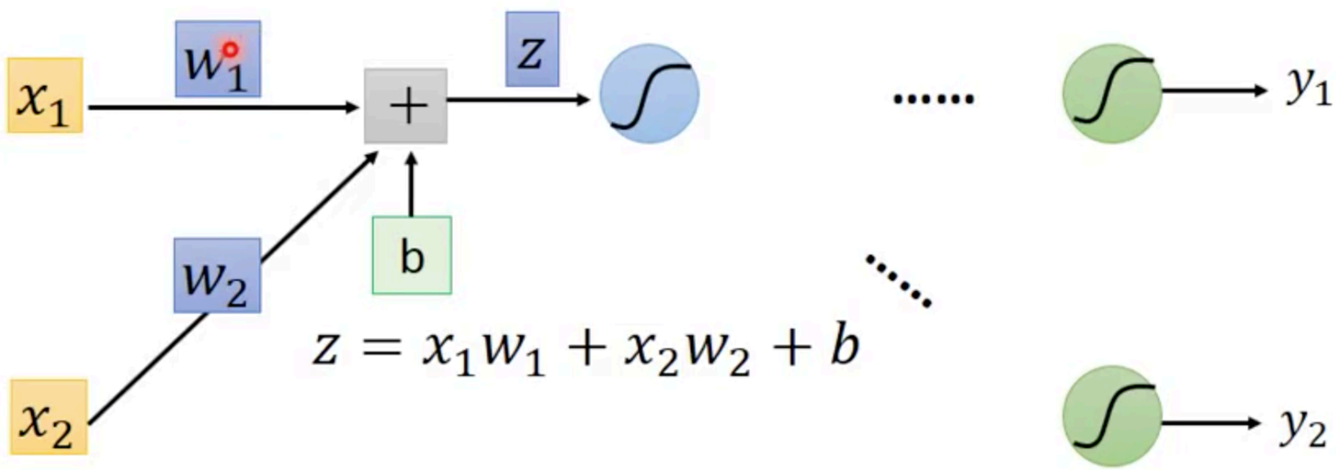
$$z = x_1 w_1 + x_2 w_2 + b$$

$\partial z/\partial w_1 =?$ x_1

$\partial z/\partial w_2 =?$ x_2

Backpropagation – Forward pass

Compute $\partial z / \partial w$ for all parameters



$$z = x_1 w_1 + x_2 w_2 + b$$

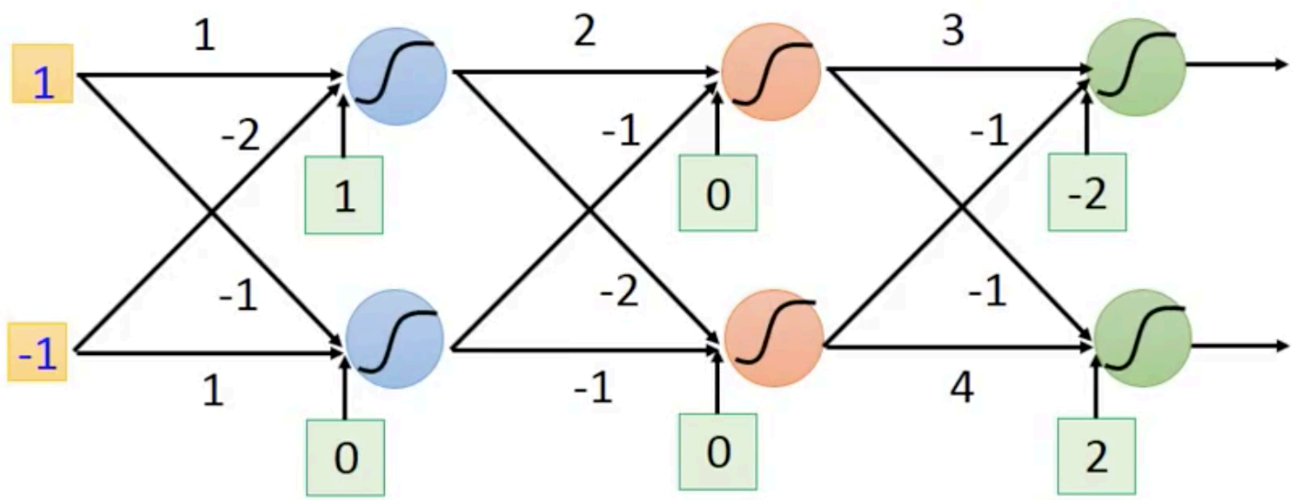
$\partial z / \partial w_1 = ?\ x_1$

$\partial z / \partial w_2 = ?\ x_2$

The value of the input connected by the weight

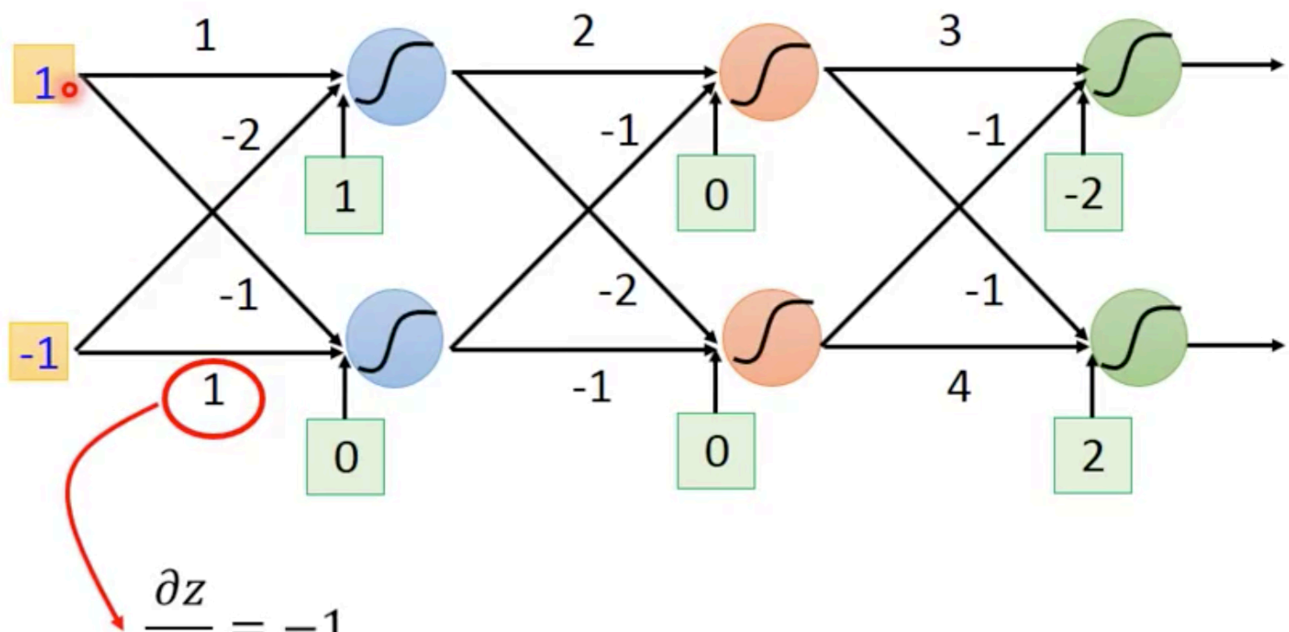
Backpropagation – Forward pass

Compute $\partial z/\partial w$ for all parameters



Backpropagation – Forward pass

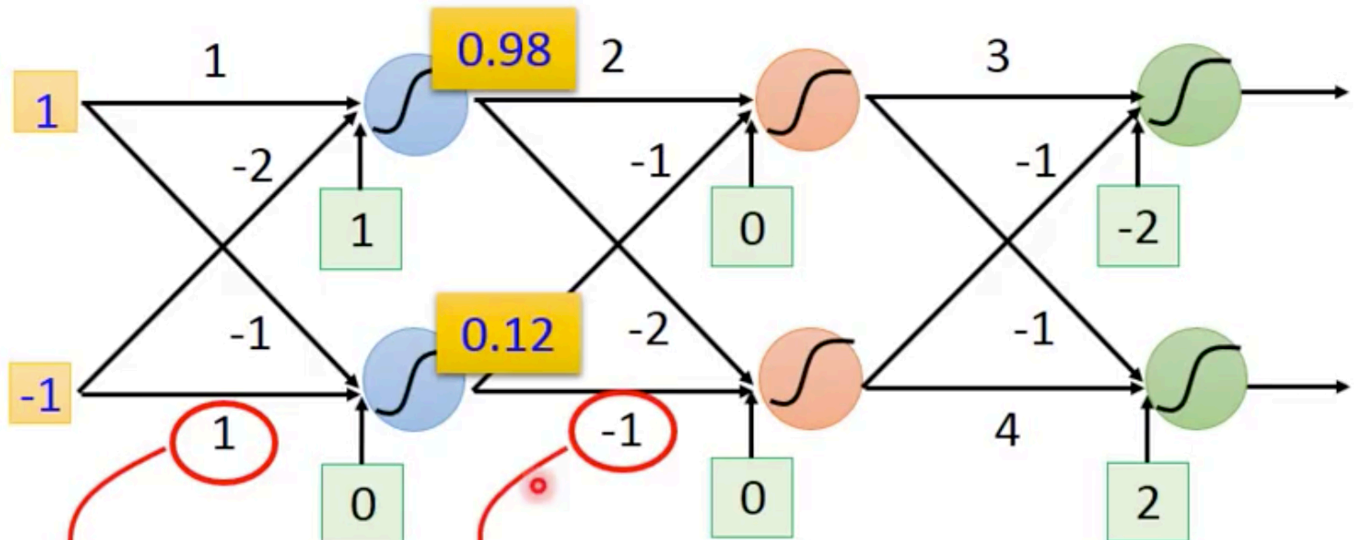
Compute $\partial z/\partial w$ for all parameters



$$\frac{\partial z}{\partial w} = -1$$

Backpropagation – Forward pass

Compute $\partial z/\partial w$ for all parameters

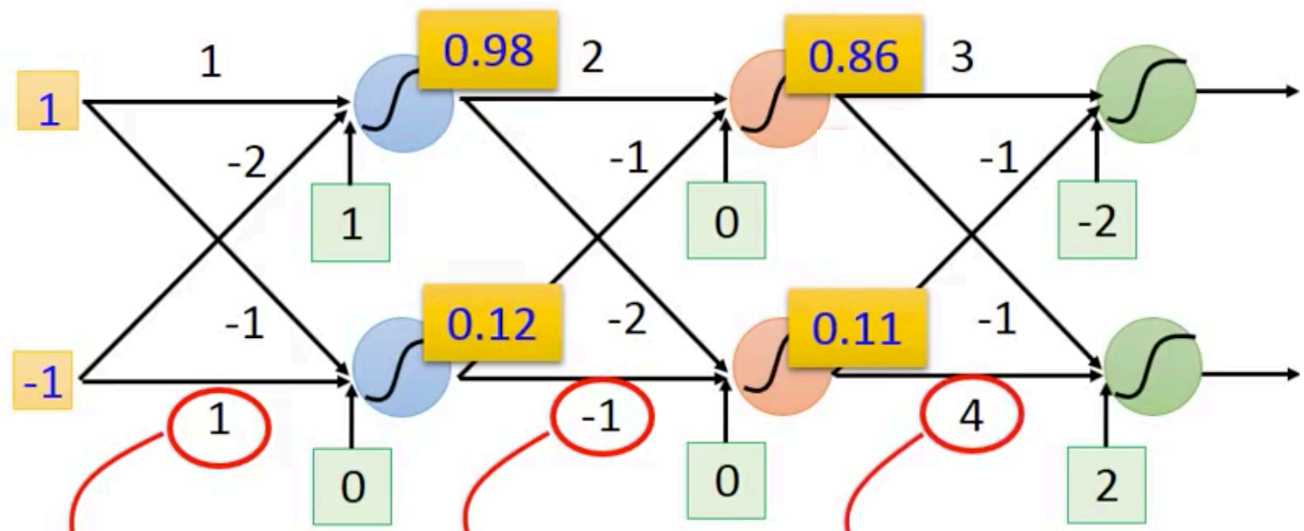


$$\frac{\partial z}{\partial w} = -1$$

$$\frac{\partial z}{\partial w} = 0.12$$

Backpropagation – Forward pass

Compute $\partial z/\partial w$ for all parameters



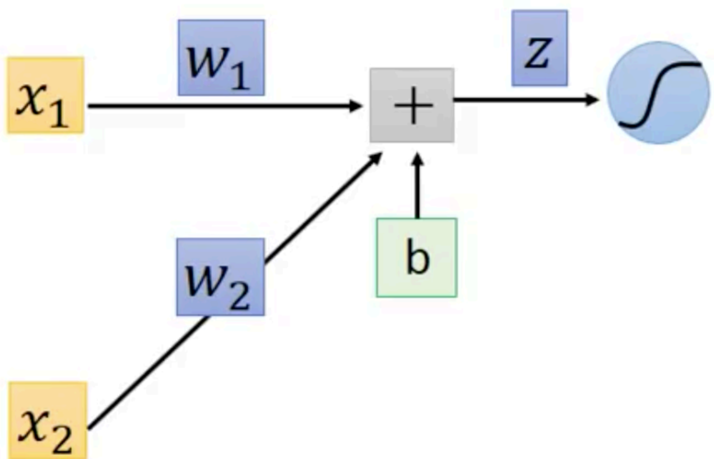
$$\frac{\partial z}{\partial w} = -1$$

$$\frac{\partial z}{\partial w} = 0.12$$

$$\frac{\partial z}{\partial w} = 0.11$$

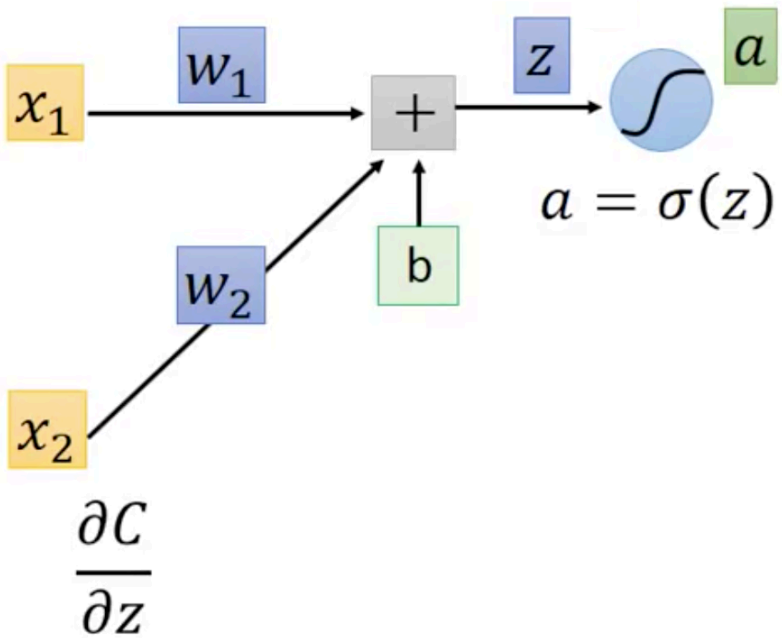
Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z



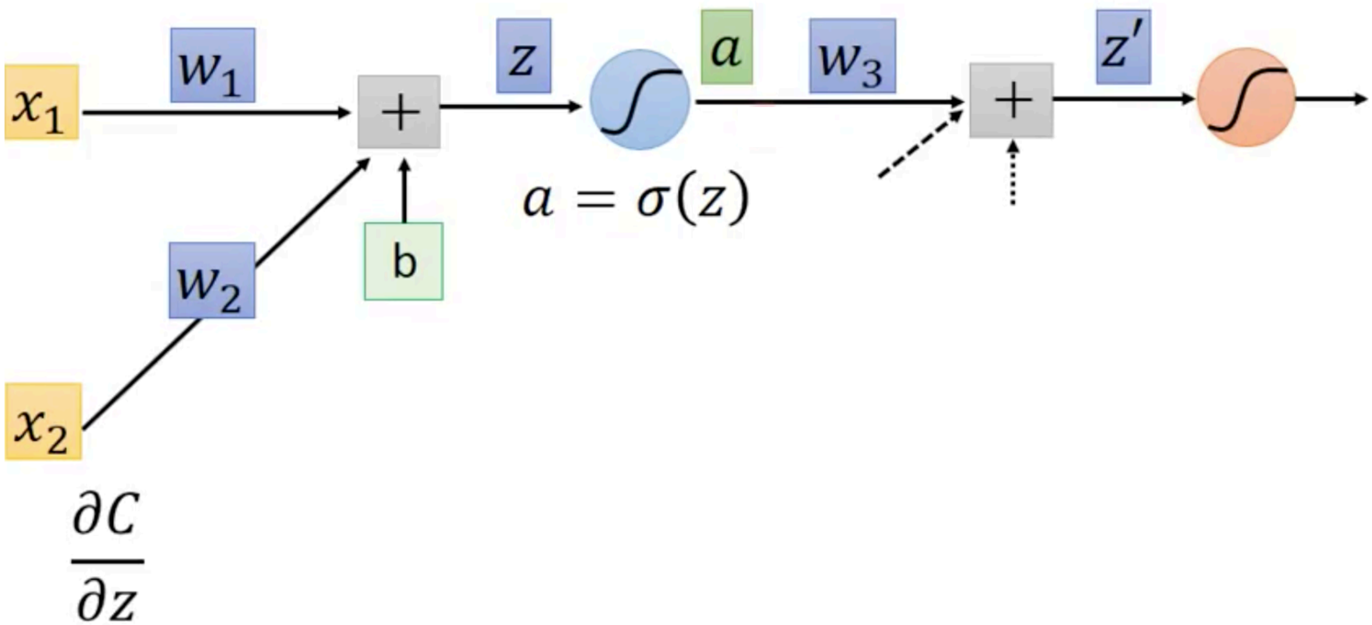
Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z



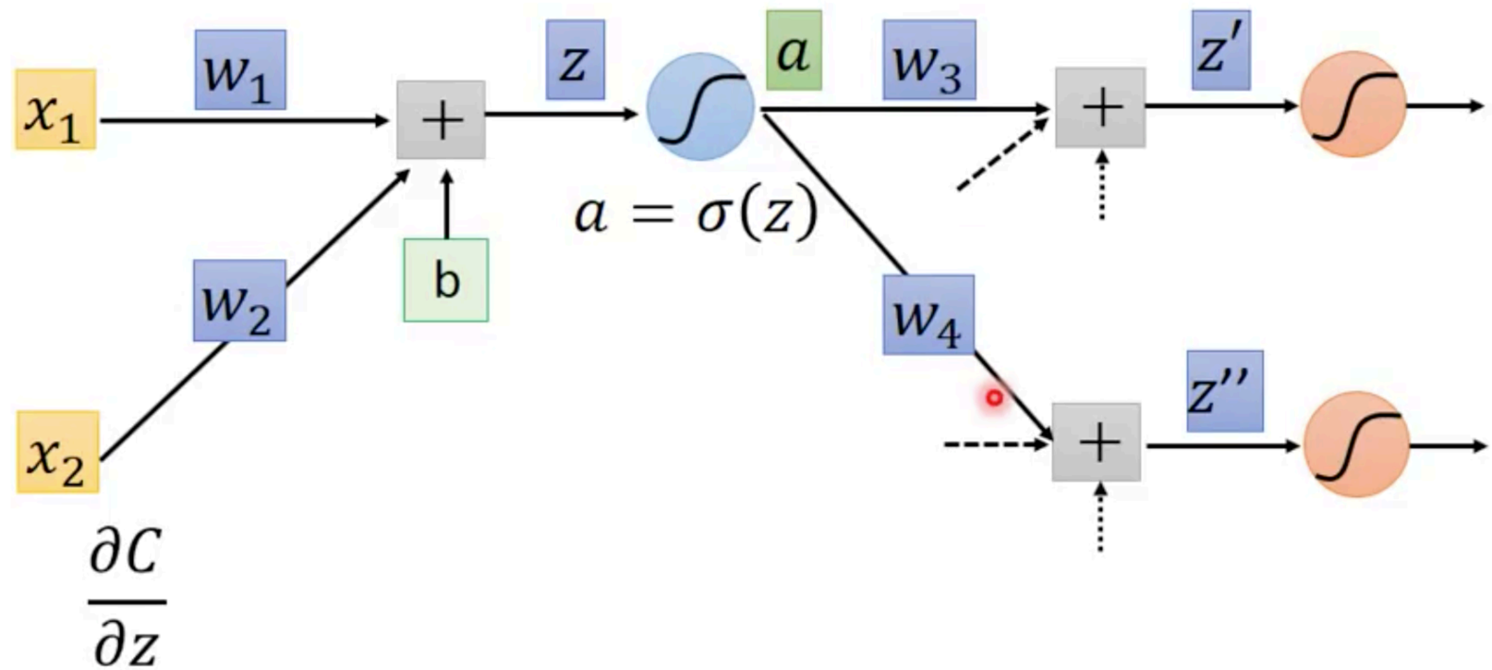
Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z



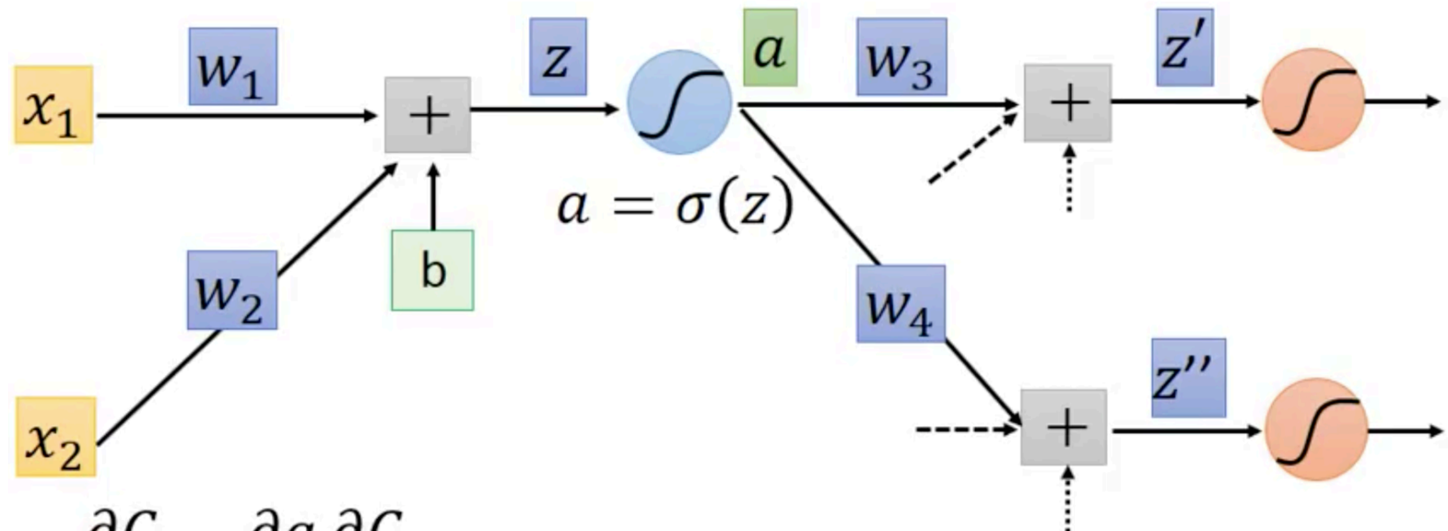
Backpropagation – Backward pass

Compute $\partial C/\partial z$ for all activation function inputs z



Backpropagation – Backward pass

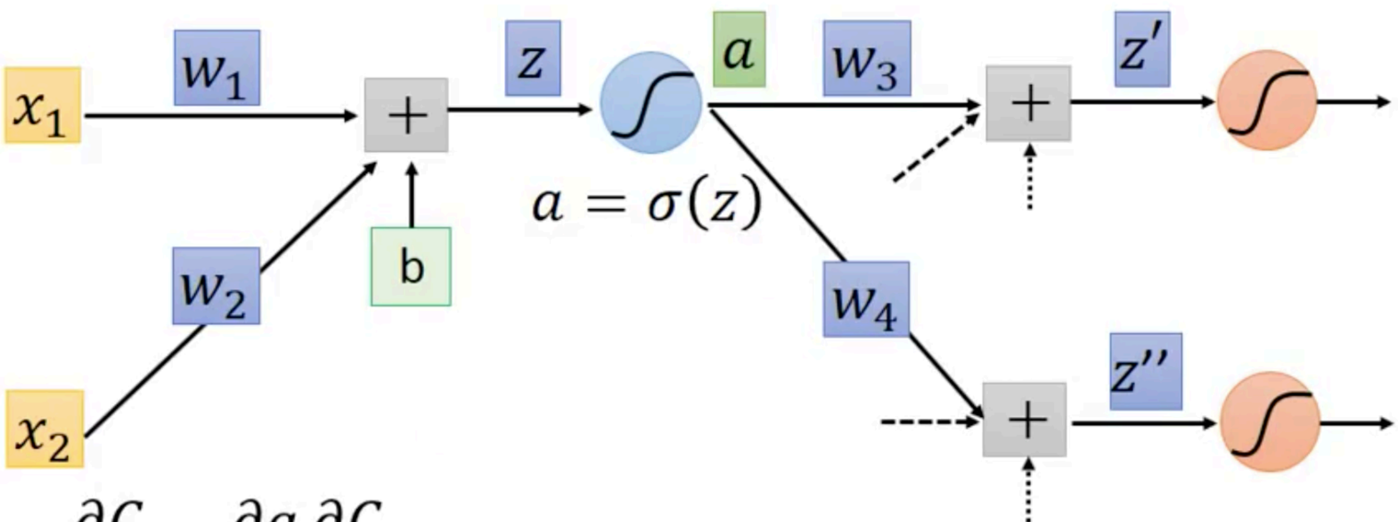
Compute $\partial C / \partial z$ for all activation function inputs z



$$\frac{\partial C}{\partial z} = \frac{\partial a}{\partial z} \frac{\partial C}{\partial a}$$

Backpropagation – Backward pass

Compute $\partial C/\partial z$ for all activation function inputs z

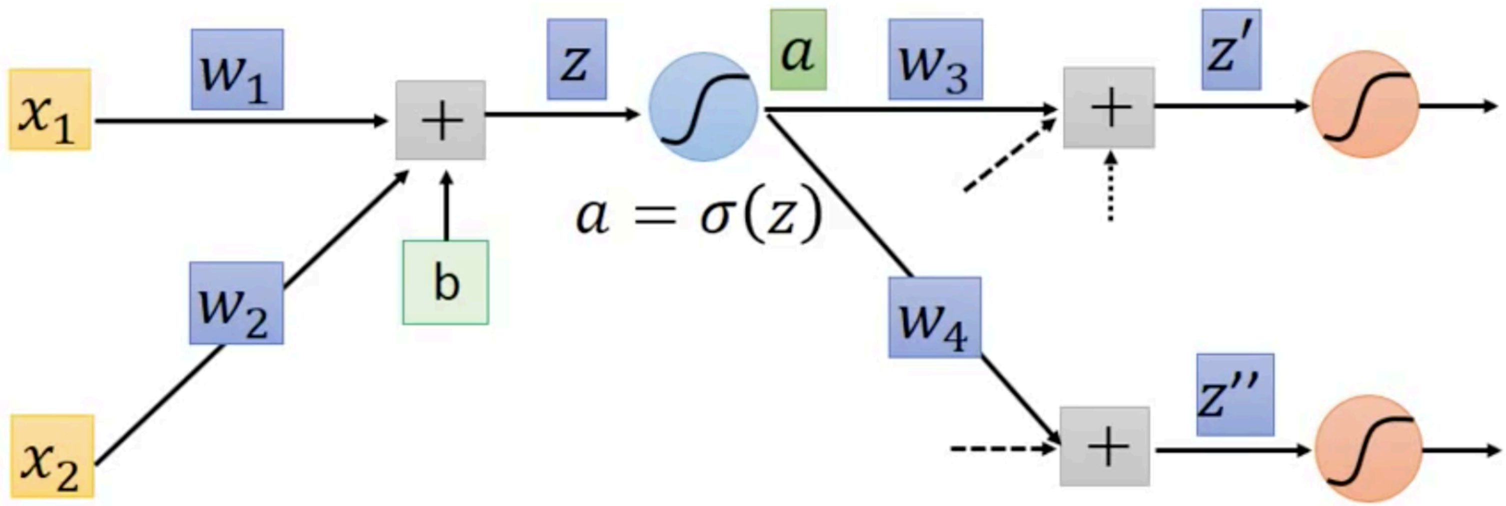


$$\frac{\partial C}{\partial z} = \frac{\partial a}{\partial z}\frac{\partial C}{\partial a}$$

→ $\sigma'(z)$

Backpropagation – Backward pass

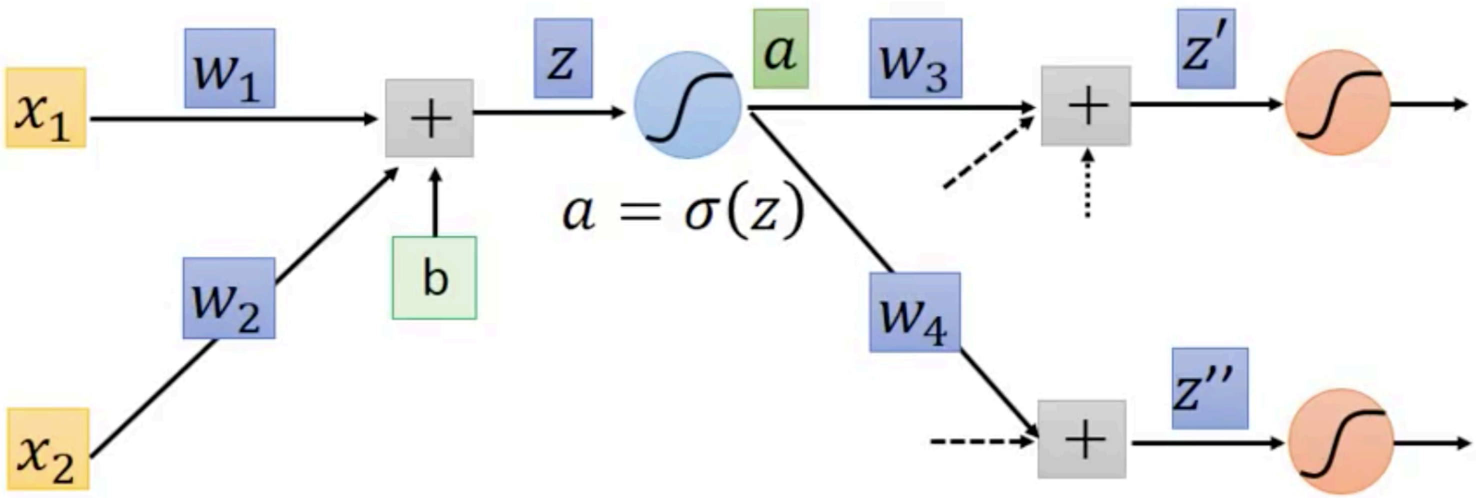
Compute $\partial C / \partial z$ for all activation function inputs z



$$\frac{\partial C}{\partial z} = \frac{\partial a}{\partial z} \frac{\partial C}{\partial a}$$

Backpropagation – Backward pass

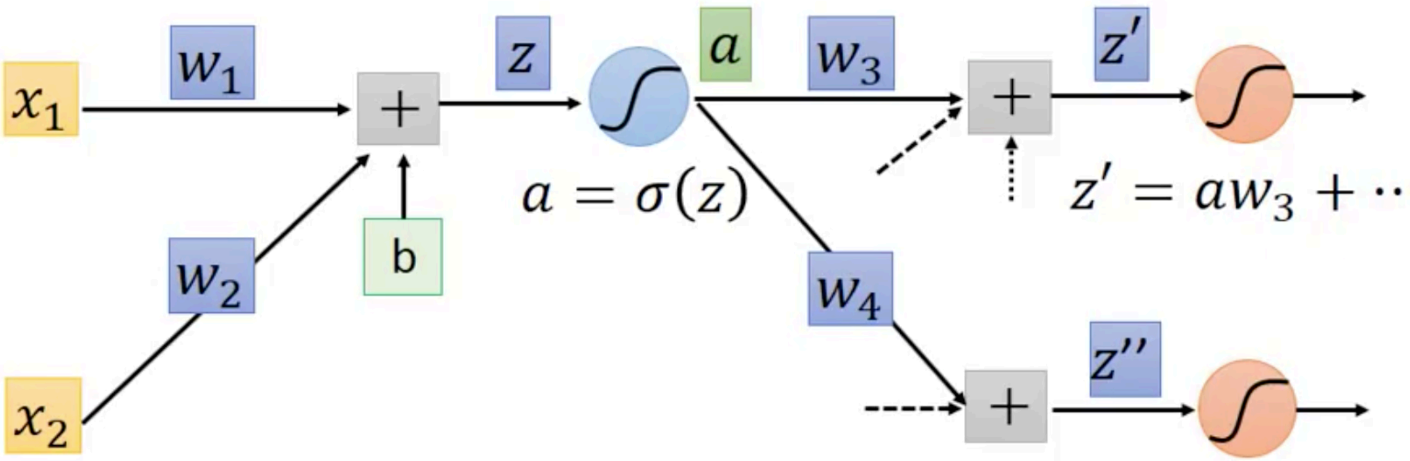
Compute $\partial C / \partial z$ for all activation function inputs z



$$\frac{\partial C}{\partial z} = \frac{\partial a}{\partial z} \frac{\partial C}{\partial a} \qquad \frac{\partial C}{\partial a} = \frac{\partial z'}{\partial a} \frac{\partial C}{\partial z'} + \frac{\partial z''}{\partial a} \frac{\partial C}{\partial z''}$$ (Chain rule)

Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z

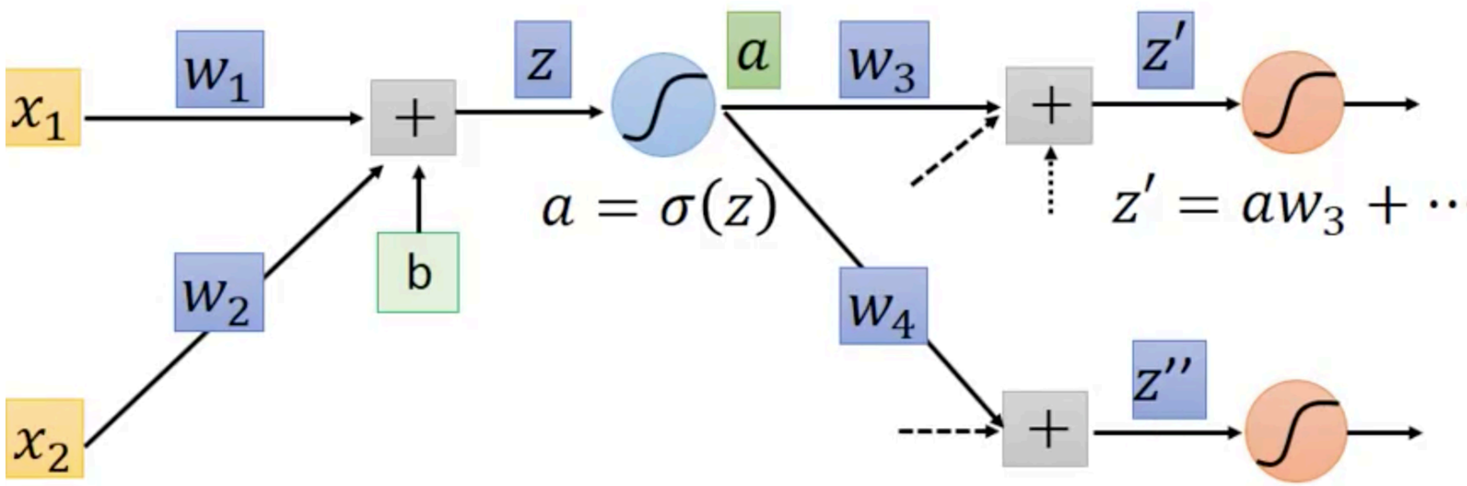


$$\frac{\partial C}{\partial z} = \frac{\partial a}{\partial z}\frac{\partial C}{\partial a} \qquad \frac{\partial C}{\partial a} = \frac{\partial z'}{\partial a}\frac{\partial C}{\partial z'} + \frac{\partial z''}{\partial a}\frac{\partial C}{\partial z''}$$ (Chain rule)

w_3

Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z

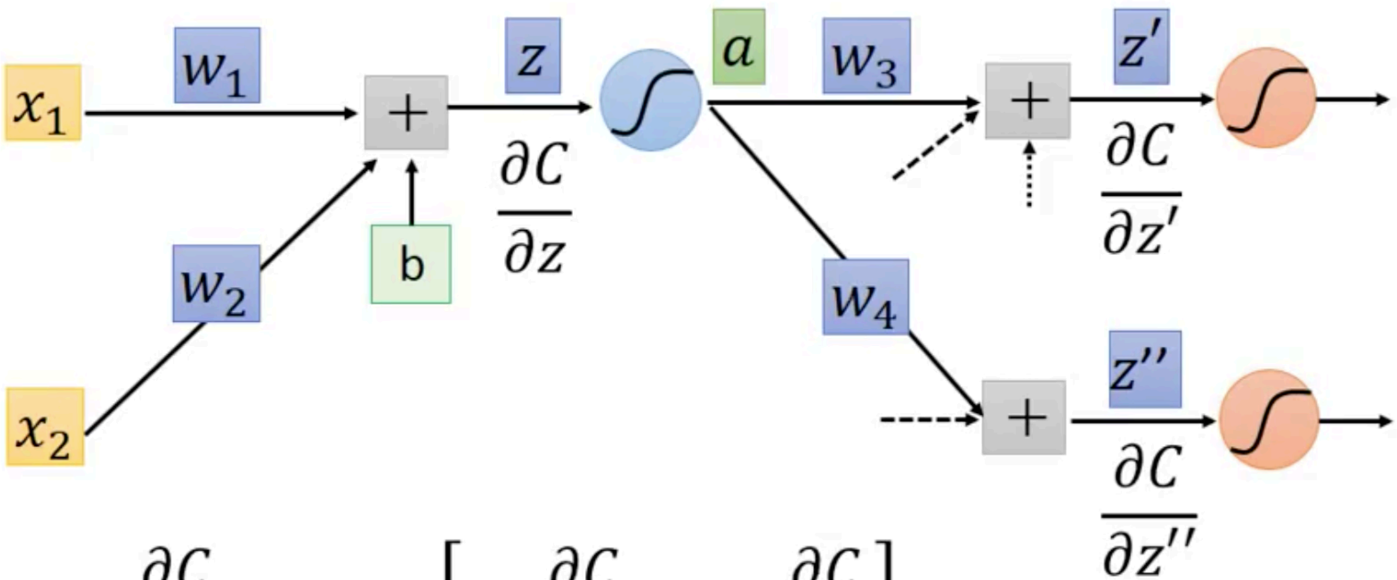


$$\frac{\partial C}{\partial z} = \frac{\partial a}{\partial z}\frac{\partial C}{\partial a} \qquad \frac{\partial C}{\partial a} = \frac{\partial z'}{\partial a}\frac{\partial C}{\partial z'} + \frac{\partial z''}{\partial a}\frac{\partial C}{\partial z''}$$ (Chain rule)

w_3 w_4

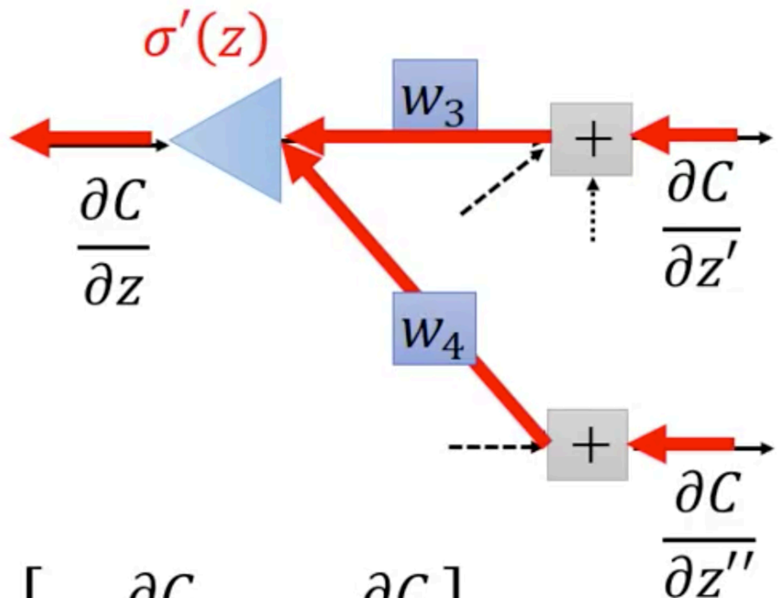
Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z



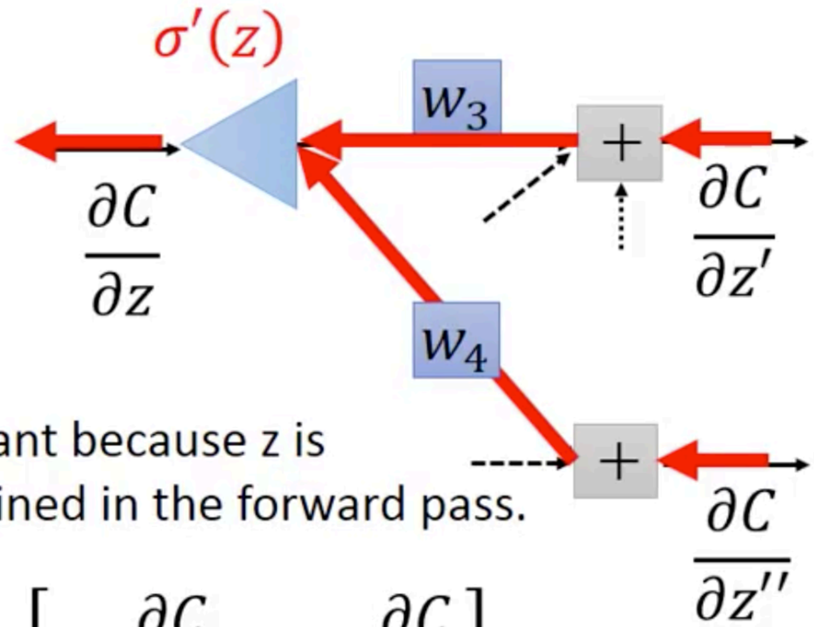
$$\frac{\partial C}{\partial z} = \sigma'(z)\left[w_3 \frac{\partial C}{\partial z'} + w_4 \frac{\partial C}{\partial z''}\right]$$

Backpropagation – Backward pass



$$\frac{\partial C}{\partial z} = \sigma'(z)\left[w_3 \frac{\partial C}{\partial z'} + w_4 \frac{\partial C}{\partial z''}\right]$$

Backpropagation – Backward pass

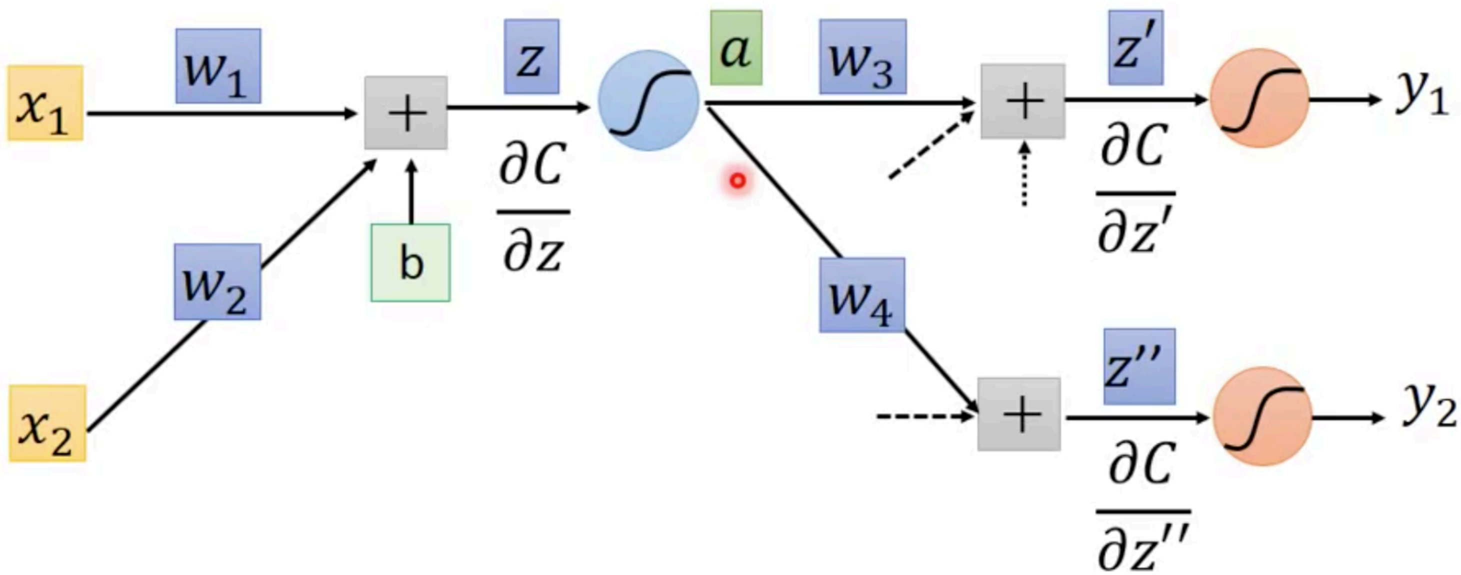


$\sigma'(z)$ is a constant because z is already determined in the forward pass.

$$\frac{\partial C}{\partial z} = \sigma'(z)\left[w_3 \frac{\partial C}{\partial z'} + w_4 \frac{\partial C}{\partial z''}\right]$$

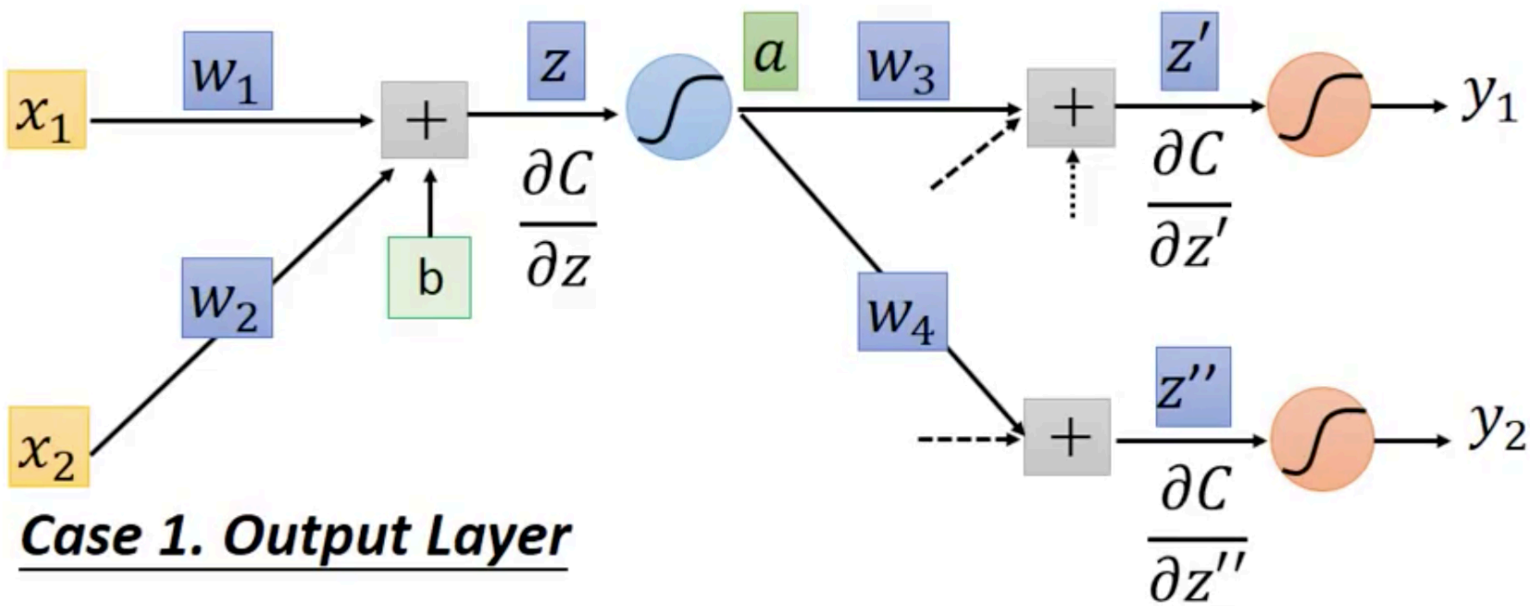
Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z



Backpropagation – Backward pass

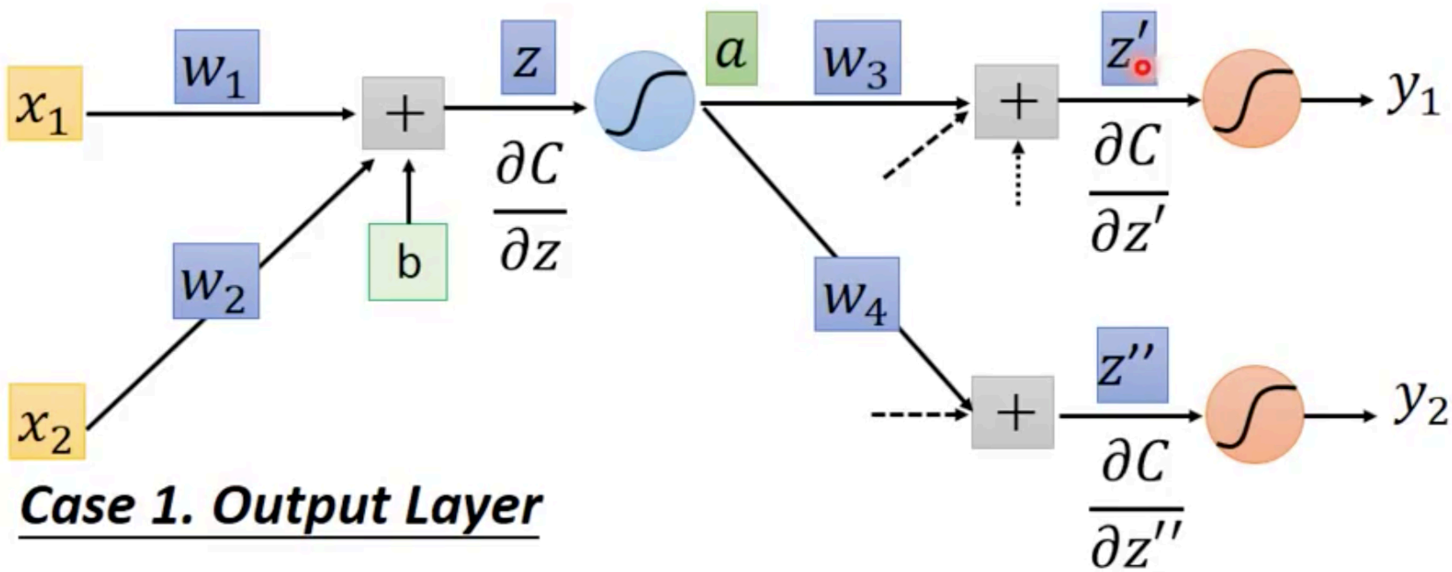
Compute $\partial C/\partial z$ for all activation function inputs z



Case 1. Output Layer

Backpropagation – Backward pass

Compute $\partial C/\partial z$ for all activation function inputs z

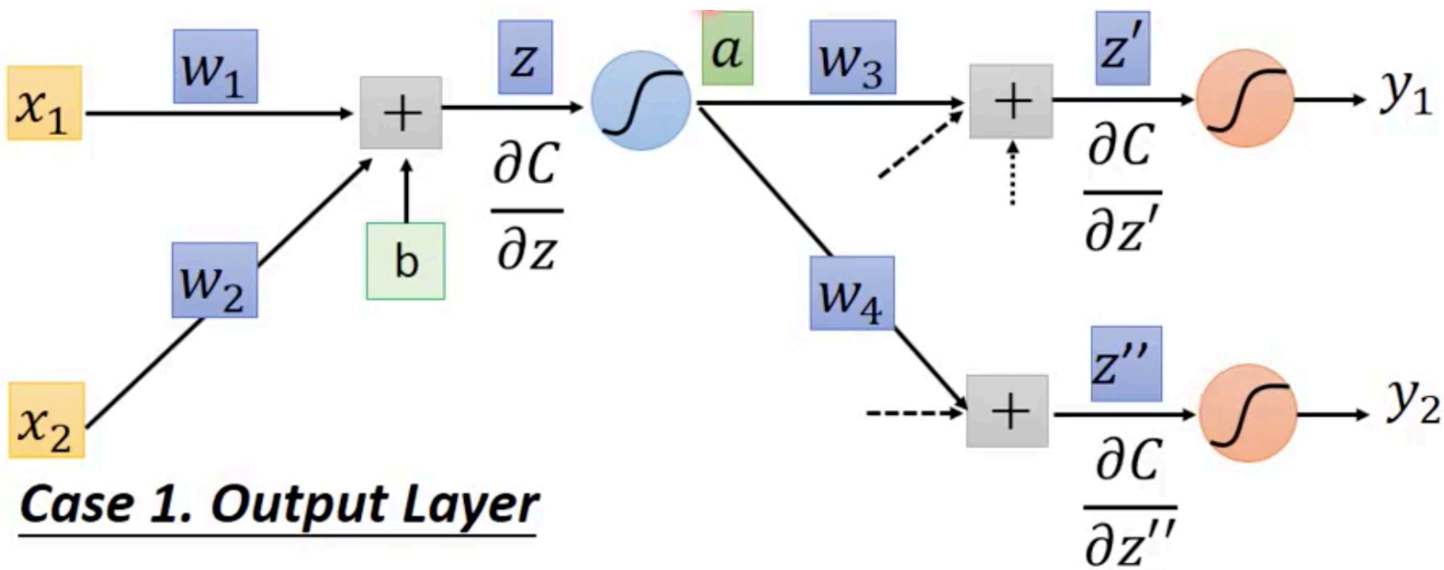


Case 1. Output Layer

$$\frac{\partial C}{\partial z'} = \frac{\partial y_1}{\partial z'} \frac{\partial C}{\partial y_1}$$

Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z



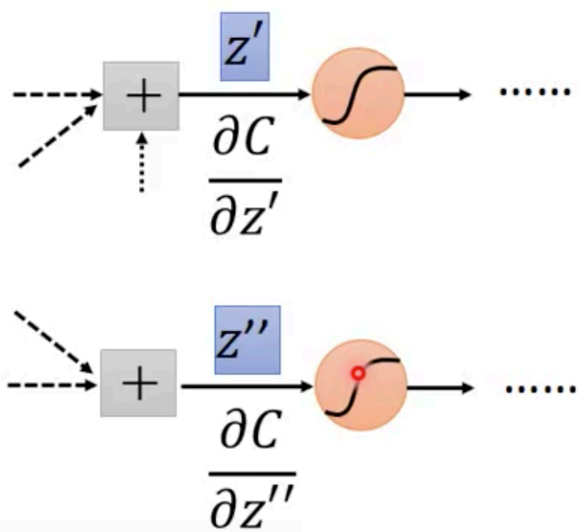
Case 1. Output Layer

$$\frac{\partial C}{\partial z'} = \frac{\partial y_1}{\partial z'} \frac{\partial C}{\partial y_1} \qquad \frac{\partial C}{\partial z''} = \frac{\partial y_2}{\partial z''} \frac{\partial C}{\partial y_2}$$

Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z

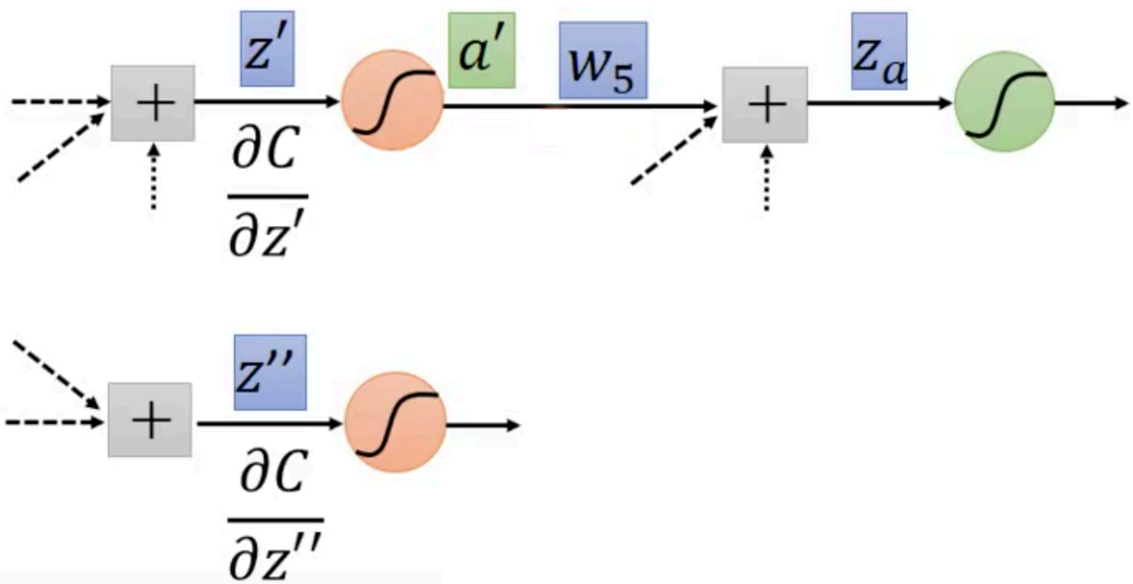
<u>Case 2. Not Output Layer</u>



Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z

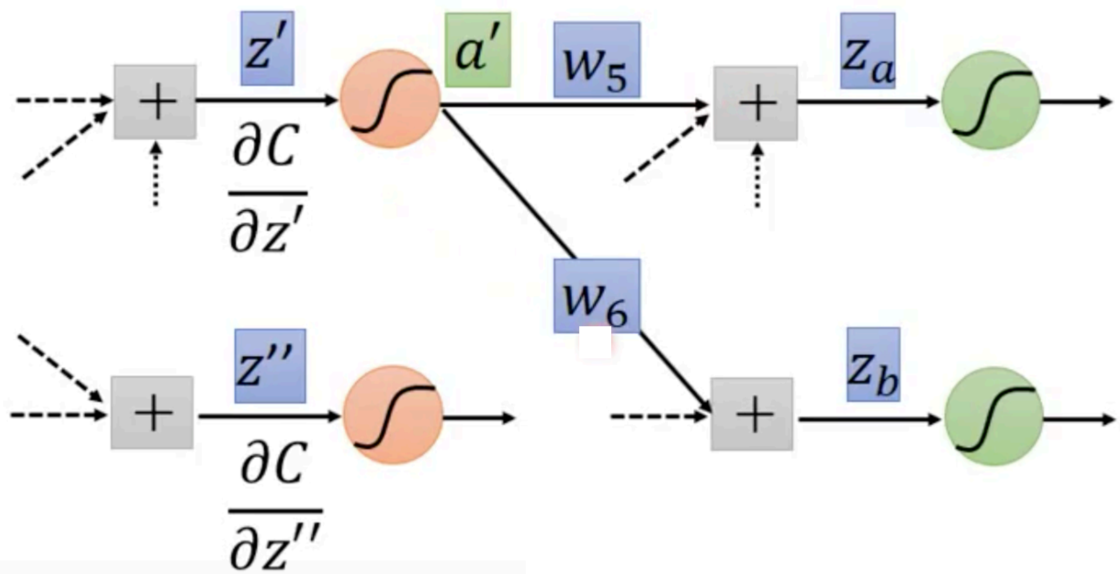
<u>Case 2. Not Output Layer</u>



Backpropagation – Backward pass

Compute $\partial C / \partial z$ for all activation function inputs z

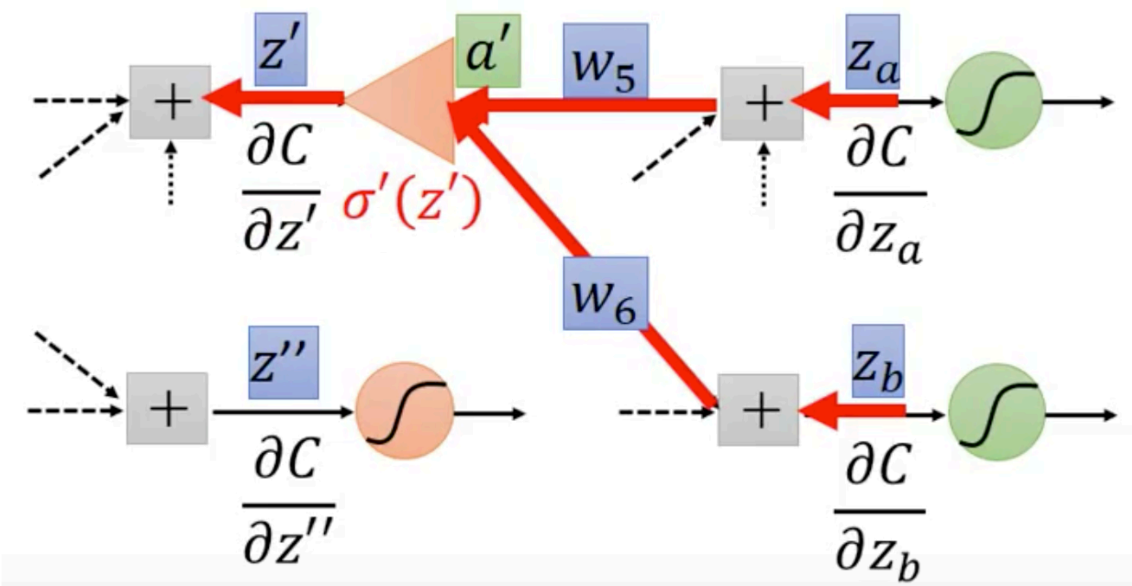
Case 2. Not Output Layer



Backpropagation – Backward pass

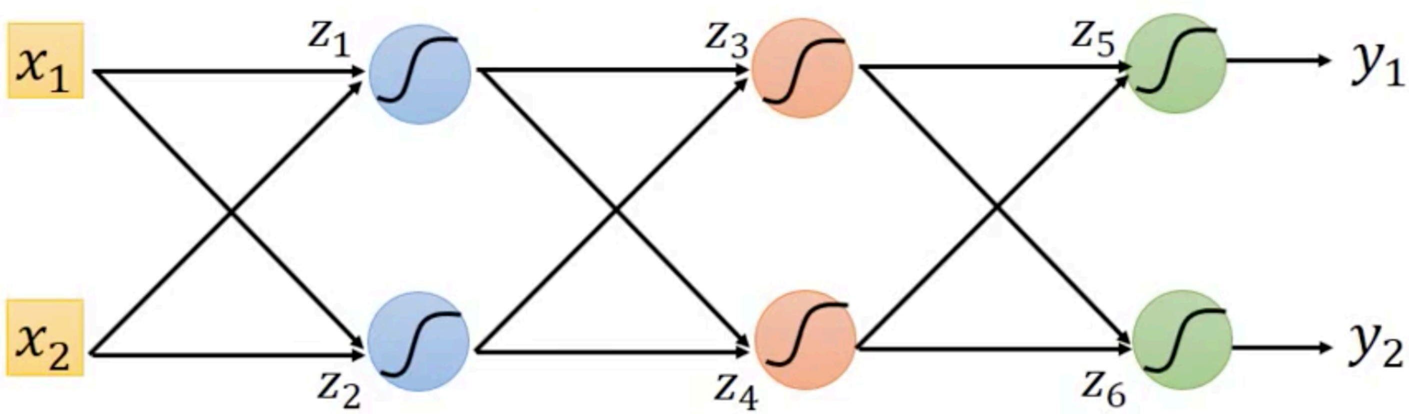
Compute $\partial C / \partial z$ for all activation function inputs z

<u>Case 2. Not Output Layer</u>



Backpropagation – Backward Pass

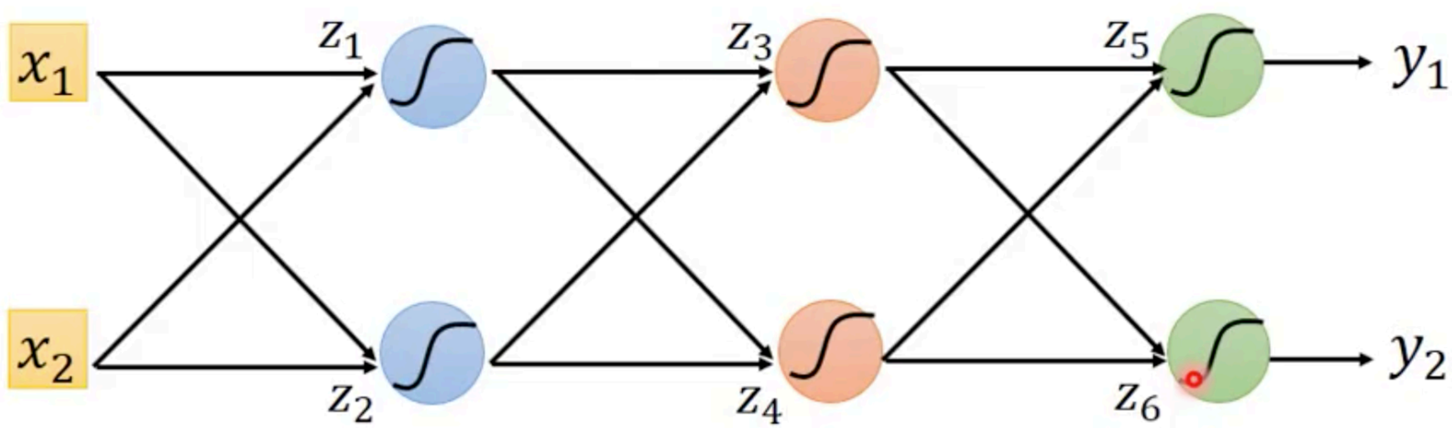
Compute $\partial C / \partial z$ for all activation function inputs z



Backpropagation – Backward Pass

Compute $\partial C / \partial z$ for all activation function inputs z

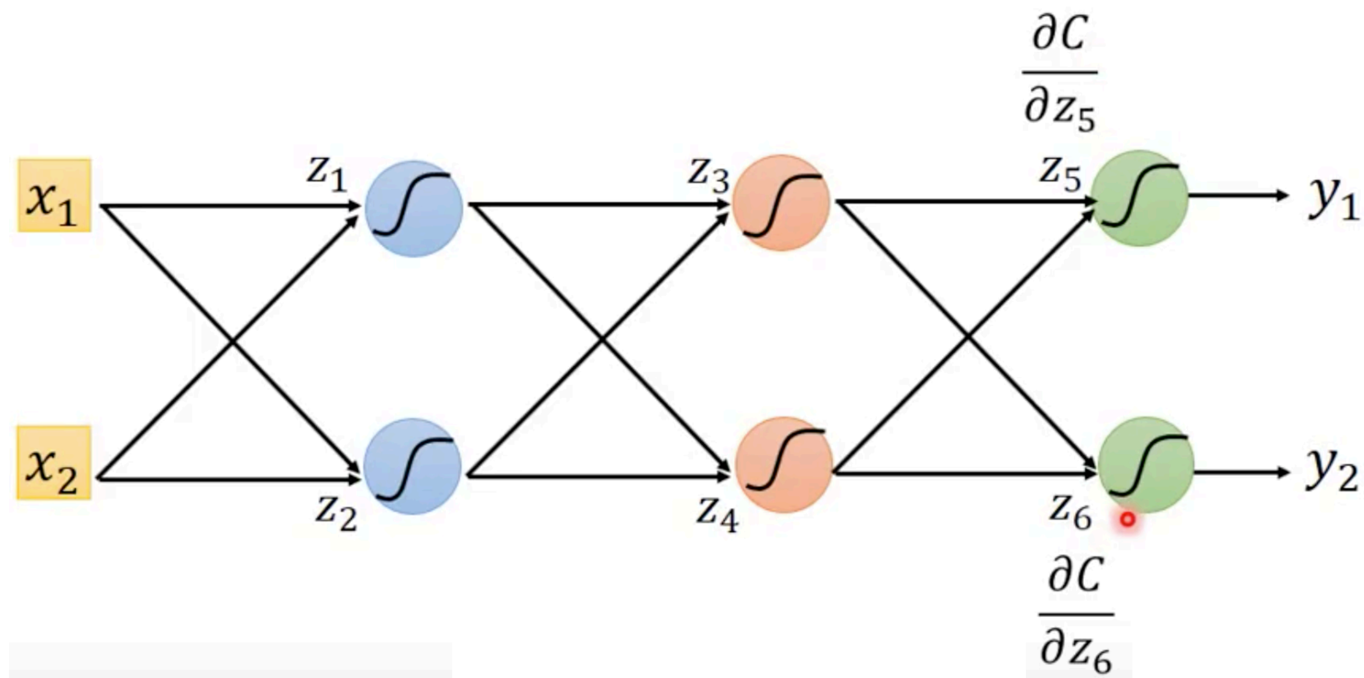
Compute $\partial C / \partial z$ from the output layer



Backpropagation – Backward Pass

Compute $\partial C / \partial z$ for all activation function inputs z

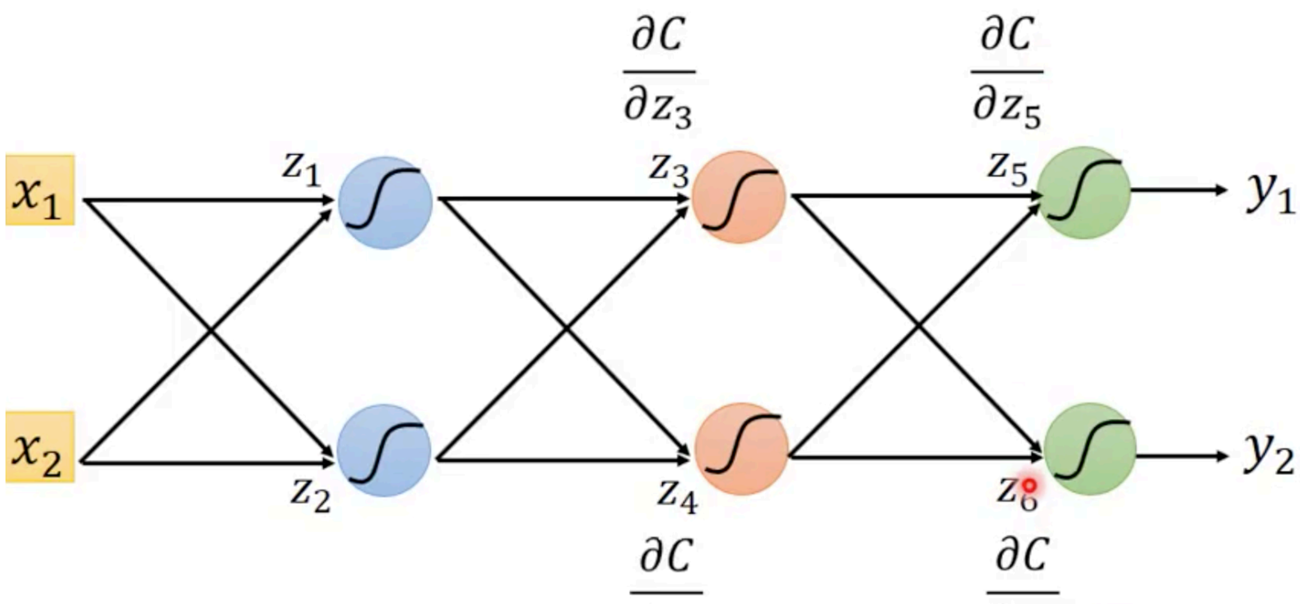
Compute $\partial C / \partial z$ from the output layer



Backpropagation – Backward Pass

Compute $\partial C / \partial z$ for all activation function inputs z

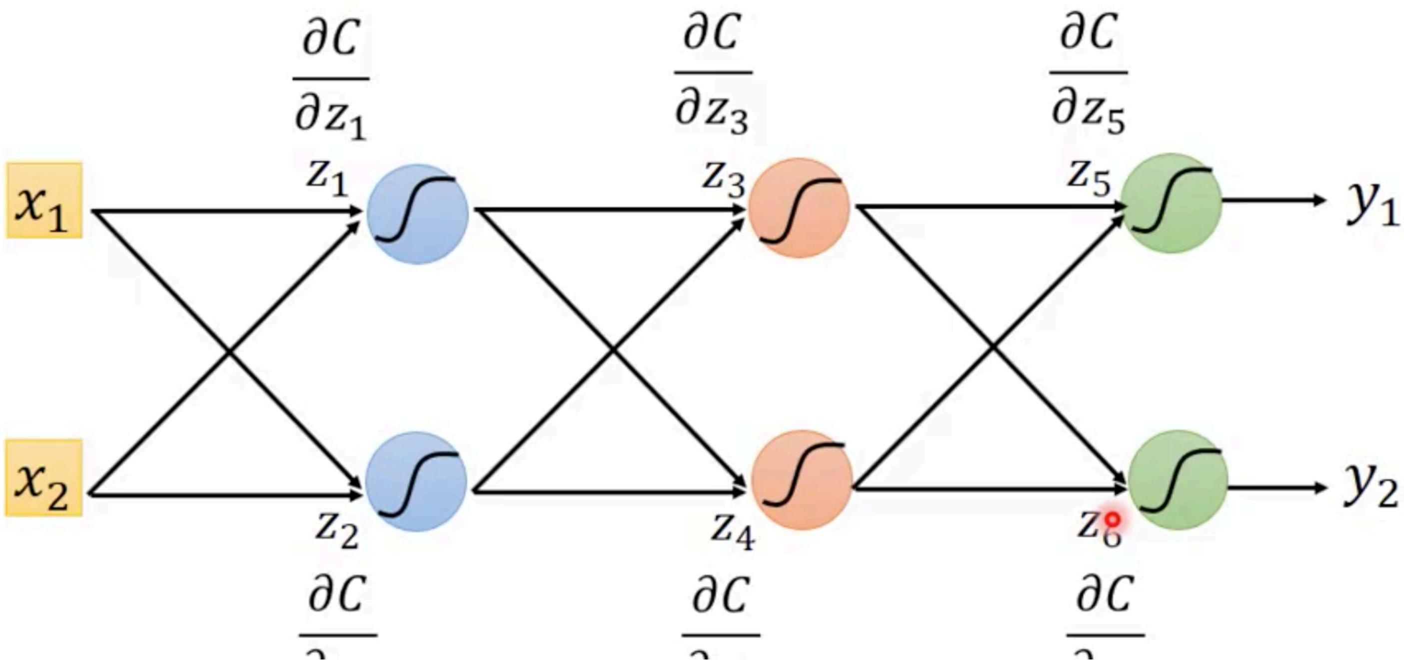
Compute $\partial C / \partial z$ from the output layer



Backpropagation – Backward Pass

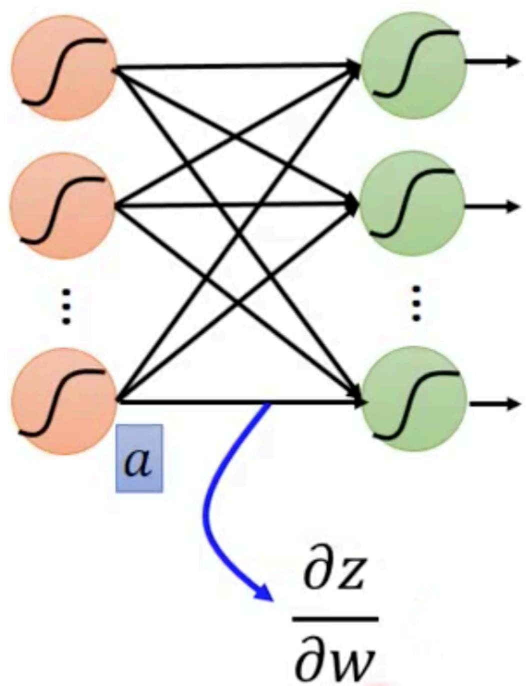
Compute $\partial C / \partial z$ for all activation function inputs z

Compute $\partial C / \partial z$ from the output layer



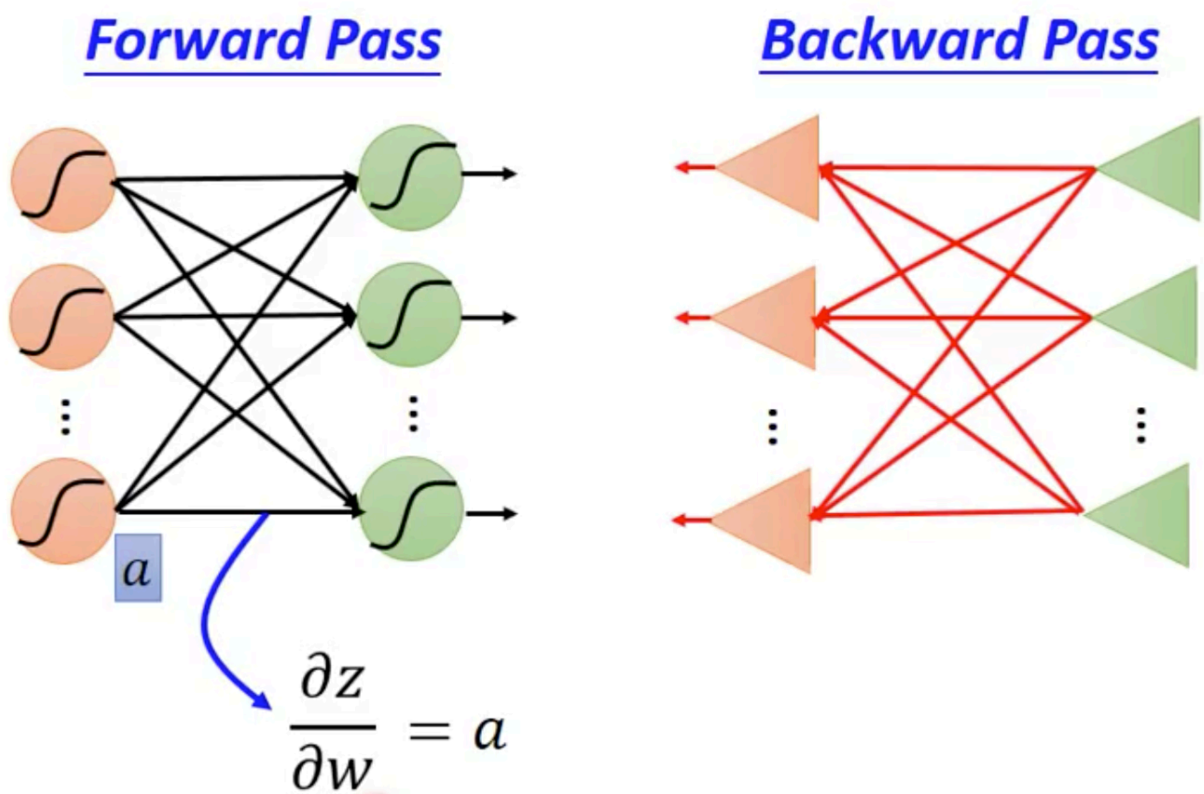
Backpropagation – Summary

Forward Pass

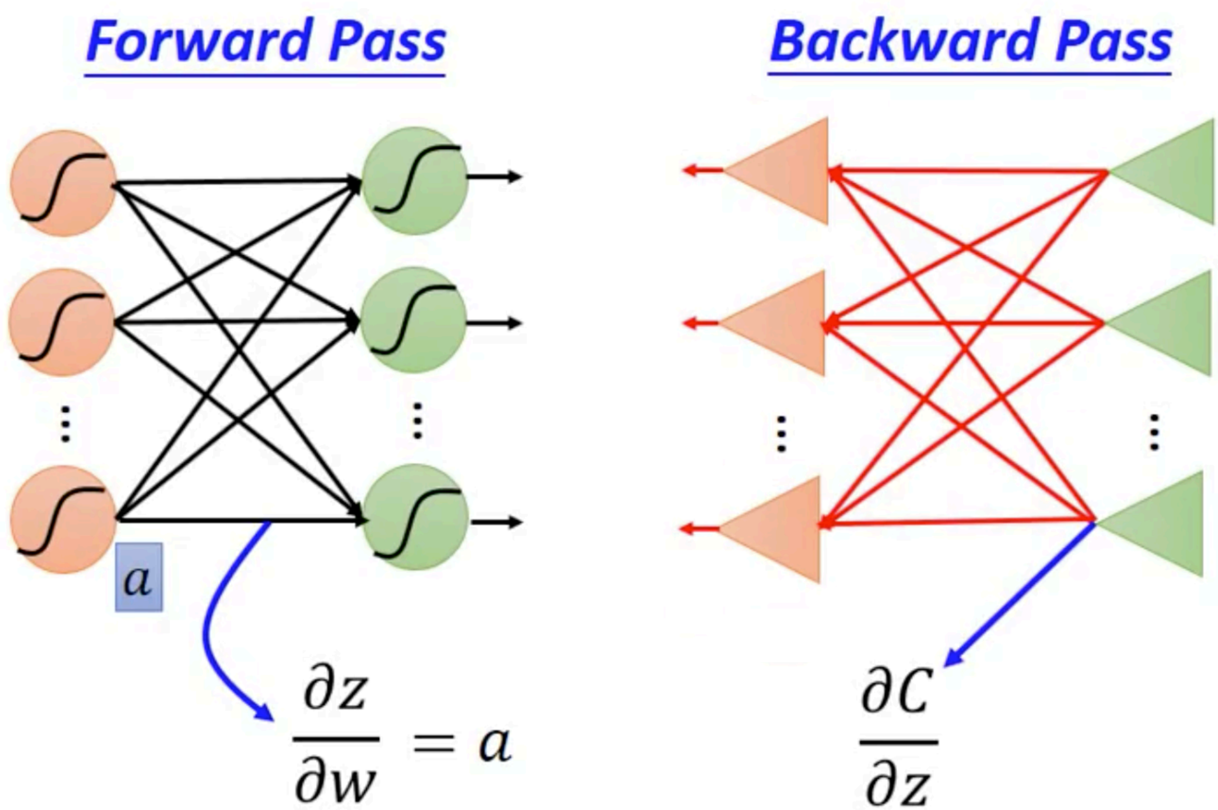


Backward Pass

Backpropagation – Summary

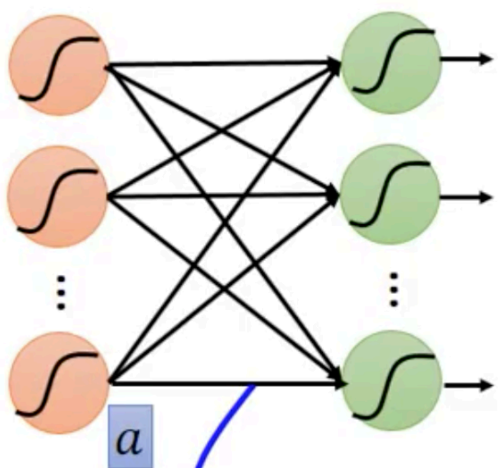


Backpropagation – Summary

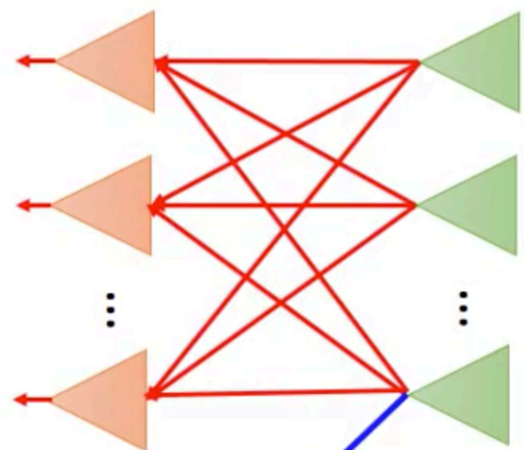


Backpropagation – Summary

Forward Pass



Backward Pass



$$\frac{\partial z}{\partial w} = a \quad \times \quad \frac{\partial C}{\partial z} \quad = \frac{\partial C}{\partial w}$$