CSCE 411 Design and Analysis of Algorithms

HW6: Solutions

Q - 26.2-6

Multi-source Multi-node maximum flow problem

Suppose we have a flow network G that has multiple sources $S_1, S_2, ..., S_n$ and multiple sinks $T_1, T_2, ..., T_m$. Each source S_i has a set of outgoing edges $E_i (i = 1, ..., n)$, and each sink T_j has a set of incoming edges $F_j (i = 1, ..., m)$. The max-flow problem on G can be reduced to the original max-flow problem by constructing a network G' from G as follows:

- We introduce two additional vertices S and T
- We construct n edges $e_1, e_2, ..., e_n$, each of them going from S to $S_1, ..., S_n$.
- We construct m edges $f_1, f_2, ..., f_m$ each of them going from $T_1, ..., T_m$ to T.
- For each e_i from S to S_i, e_i has capacity equal to the sum of the capacities of all edges in E_i.
- For each f_j from T_j to T, f_j has capacity equal to the sum of the capacities of all edges in F_j.
- S is the single source of G' and T is the single sink of G'.
- The original $S_1, ..., S_n$ and $T_1, ..., T_m$ are treated as transshipment nodes. We can apply Ford Fulkerson algorithm with single source and single sink in network G'

Q - 26.2-11

Edge Connectivity

Construct a directed graph G' from G by replacing each edge u, v in G by two directed edges (u, v) and (v, u) in G'. Let $g(u, v)$ be the maximum flow value form u to v through G' with all edge capacities equal to one. Pick an arbitrary node u and compute $g(u, v)$ for all $v \neq u$. We claim that the edge connectivity equals $c^* = min_{v \neq u} g(u, v)$. Therefore the edge connectivity can be computed by running maximum-flow algorithm $|V| - 1$ times on the flow networks each having $|V|$ veritces and $2|E|$ edges.

Suppose k is the edge connectivity of the graph and Q is the set of k edges such that removal of Q will disconnect the graph in two non-empty subgraphs G_1 and G_2. Without loss of generality assume the node $u \in G_1$. Let w be a node in G_2. Since $u \neq v$, the value $g(u, w)$ will be computed by the algorithm. By the max-flow min-cut theorem, $g(u, w)$ equals the min-cut size between the pair (u, w), which is at most k since Q disconnect u and w. Therefore, we have

$$c^* \leq g(u, w) \leq k \tag{1}$$

But c^* cannot be smaller than k since that would imply a cut set of size smaller that k, contradicting the fact that k is the edge connectivity. Therefore $c^* = k$ and the algorithm returns the edge connectivity of the graph correctly.